

ENDOSCOPIC LIFTING OF SIMPLE SUPERCUSPIDAL REPRESENTATIONS OF SO_{2n+1} TO GL_{2n}

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ABSTRACT. We compute the characters of simple supercuspidal representations of twisted $\mathrm{GL}_{2n}(F)$ and standard $\mathrm{SO}_{2n+1}(F)$ for a p -adic field F . Comparing them by the endoscopic character relation, we determine the liftings of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$, under the assumption that p is not equal to 2.

1. INTRODUCTION

Let G be a connected reductive group over a p -adic field F . Let $\Pi(G)$ be the set of equivalence classes of irreducible smooth representations of $G(F)$ and $\Phi(G)$ be the set of equivalence classes of L -parameters of G . Here an L -parameter of G is a homomorphism from the Weil-Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ to the L -group ${}^L G = \widehat{G} \rtimes W_F$ of G . Then the conjectural local Langlands correspondence predicts that there exists a natural map from $\Pi(G)$ to $\Phi(G)$ with finite fibers (L -packets).

For $G = \mathrm{GL}_n$, it was established by Harris-Taylor [HT01] and Henniart [Hen00]. In this case, the map from $\Pi(G)$ to $\Phi(G)$ is bijective.

For quasi-split symplectic and orthogonal groups, the correspondence was established recently by Arthur in his book [Art13], under the assumption of the stabilization of twisted trace formulas for general linear groups and even orthogonal groups. He characterized the L -packets for L -parameters of those groups via the *endoscopic character relation*, and constructed them in consequence of the comparison of stable trace formulas.

We explain the endoscopic character relation. For simplicity, we consider the special orthogonal group SO_{2n+1} , which is a main object in this paper. This group is a twisted endoscopic group for GL_{2n} , and there exists a natural L -embedding

$$\iota: {}^L \mathrm{SO}_{2n+1} = \mathrm{Sp}_{2n}(\mathbb{C}) \times W_F \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_F = {}^L \mathrm{GL}_{2n}.$$

Hence we can regard an L -parameter ϕ' of SO_{2n+1} as an L -parameter ϕ of GL_{2n} by composing ι . By using the local Langlands correspondence for GL_{2n} , we can get the representation π of $\mathrm{GL}_{2n}(F)$ corresponding to ϕ .

$$\begin{array}{ccc} \Pi(\mathrm{GL}_{2n}) \ni \pi & \begin{array}{c} \text{LLC for } \mathrm{GL}_{2n} \\ \swarrow \sim \searrow \end{array} & W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi} {}^L \mathrm{GL}_{2n} \\ & & \searrow \phi' \downarrow \iota \\ \Pi(\mathrm{SO}_{2n+1}) \supset \Pi_{\phi'} & \begin{array}{c} \text{LLC for } \mathrm{SO}_{2n+1} \\ \swarrow \sim \searrow \end{array} & {}^L \mathrm{SO}_{2n+1} \end{array}$$

In this situation, π is called the *lifting* of $\Pi_{\phi'}$. Then the endoscopic character relation is an equality between the characters of representations in $\Pi_{\phi'}$ and the twisted character of π .

Therefore the local Langlands correspondence for SO_{2n+1} is reduced to determine the liftings of representations of $\mathrm{SO}_{2n+1}(F)$, and it is important to compute the characters of representations for this.

In this paper, we consider this problem for so-called *simple supercuspidal* representations of $\mathrm{SO}_{2n+1}(F)$. These representations are supercuspidal representations which are obtained by compact induction of characters of the pro-unipotent radicals of Iwahori subgroups (see Section 2 for details). They were introduced in [GR10] and [RY14], and have been studied in the context of finding an explicit description of the local Langlands correspondence.

We explain our main result. The set of equivalence classes of self-dual simple supercuspidal representations of $\mathrm{GL}_{2n}(F)$ with trivial central characters is (non-canonically) parametrized by the finite set $k^\times \times \{\pm 1\}$. On the other hand, the set of equivalence classes of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ is also parametrized by $k^\times \times \{\pm 1\}$. Under these parametrization, for $(a, \zeta) \in k^\times \times \{\pm 1\}$ (resp. $(b, \xi) \in k^\times \times \{\pm 1\}$), we denote by $\pi_{a, \zeta}$ (resp. $\pi'_{b, \xi}$) the corresponding simple supercuspidal representations. Then our main theorem on the liftings of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ is stated as follows:

Theorem 1.1 (Theorem 5.16). *We assume $p \neq 2$. Let $b \in k^\times$ and $\xi \in \{\pm 1\}$.*

- (1) *The L -packet containing the simple supercuspidal representation $\pi'_{b, \xi}$ of $\mathrm{SO}_{2n+1}(F)$ is a singleton. In particular, the character of $\pi'_{b, \xi}$ is stable.*
- (2) *The lifting of the simple supercuspidal representation $\pi'_{b, \xi}$ of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ is again simple supercuspidal, and given by $\pi_{2b, \xi}$.*

We remark that the L -parameters of simple supercuspidal representations of general linear groups have been described explicitly by the works of [BH05] and [IT15]. From Theorem 1.1, we know that the L -parameter of $\pi'_{b, \xi}$ is equal to that of $\pi_{2b, \xi}$. Therefore we get an explicit description of the L -parameters of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$.

We explain the outline of our proof. To prove Theorem 1.1, we consider the converse direction. Namely, we first take the self-dual simple supercuspidal representation $\pi = \pi_{2b, \xi}$ of $\mathrm{GL}_{2n}(F)$ for $b \in k^\times$ and $\xi \in \{\pm 1\}$, and show that the L -parameter ϕ of π factors through an L -parameter ϕ' of SO_{2n+1} , and its L -packet $\Pi_{\phi'}$ is a singleton. Then we show that the unique representation in $\Pi_{\phi'}$ is $\pi'_{b, \xi}$.

The first step follows from general results. Since the representation π is self-dual, the L -parameter ϕ of π is also self-dual by a property of the local Langlands correspondence for GL_{2n} . Hence the image of ϕ in $\widehat{\mathrm{GL}_{2n}} = \mathrm{GL}_{2n}(\mathbb{C})$ is contained in either $\mathrm{O}_{2n}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$. By a result in [Mie15], the image is in $\mathrm{Sp}_{2n}(\mathbb{C}) = \widehat{\mathrm{SO}_{2n+1}}$, hence ϕ factors through an L -parameter ϕ' of SO_{2n+1} . Then, by the local classification theorem in [Art13] and the theorem for parametrizing supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ in [Xu15], we know that the L -packet for ϕ' is a singleton consisting of a supercuspidal representation π' .

The key point of the proof is the second step to show that this representation π' is in fact simple supercuspidal. Our strategy is to compare characters of π and π' . We first compute the twisted character of the simple supercuspidal representation π for special elements, which we call *affine generic* elements. By using the twisted

character formula for supercuspidal representations, we write these character values explicitly in terms of Kloosterman sums. We remark that such a computation was already done by many people in the case of standard GL_n (for example, [IT14]), but our computation in this paper is more conceptual and valid for other groups.

Then, by the endoscopic character relation, we can express the character of π' in terms of the twisted character of π , which is already computed. From this relation, we can show that π' is either simple supercuspidal or depth-zero supercuspidal. To eliminate the possibility that π' is depth-zero supercuspidal, we next compute the character of depth-zero supercuspidal representations, and compare them. Once we know that π' is simple supercuspidal, we can show that $\pi' = \pi'_{b,\xi}$ easily by computing the characters of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ and considering the Fourier transform of Kloosterman sums, and this completes the proof.

We remark that the simple supercuspidality of the lifting of a simple supercuspidal representation π' of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ was proved in [Adr15] under the assumption that $p \geq (2+e)(2n+1)$, where e is the ramification index of F over \mathbb{Q}_p . Hence our results are new for odd primes less than $(2+e)(2n+1)$. In his proof, he uses the stability of the character of π' in [Kal15], Mœglin's result about the stability of L -packets in [Mœg14], and special arguments for discrete series representations of $\mathrm{GL}_{2n}(F)$. On the other hand, in our proof, the stability of the characters of π' is naturally deduced from Arthur's theorem.

We also remark that our method is basically valid for any other group. For example, a quasi-split unitary group $\mathrm{U}_{E/F}(N)$ is a twisted endoscopic group for $\mathrm{Res}_{E/F}(\mathrm{GL}_N)$, where E/F is a quadratic extension of p -adic fields. The endoscopic classification of representations for these groups have also been established in [Mok15], and we can apply the same argument for them. This case is in progress now.

Finally, we explain the organization of this paper. In Section 2, we review some fundamental properties about Iwahori subgroups and simple supercuspidal representations. In addition, we introduce the notion of affine genericity for elements in Iwahori subgroups, which will play important roles in a comparison of characters. In Section 3, we compute the characters of simple supercuspidal representations of twisted $\mathrm{GL}_{2n}(F)$ and standard $\mathrm{SO}_{2n+1}(F)$ for affine generic elements. In Section 4, we investigate the norm correspondence for GL_{2n} and SO_{2n+1} . The norm correspondence is used to formulate the endoscopic character relation. We determine norms of affine generic elements and compute their transfer factors. In Section 5, we first recall the endoscopic character relation in [Art13]. Then we determine the liftings of simple supercuspidal representations by combining it with the results in Sections 3 and 4. In Appendix A, we list some properties about the Gauss sum and the Kloosterman sum.

Acknowledgment. This paper is part of my master's thesis. I would like to thank my advisor Yoichi Mieda for his constant support and encouragement. He always corrected my misunderstanding and led me to the right direction. I would not have been able to write this paper without his guidance. I am grateful to Naoki Imai, Teruhisa Koshikawa and Koji Shimizu for many valuable comments and pointing out a lot of mistakes and typos in the draft version of this paper. I would like to thank Takahiro Tsushima for teaching me some techniques concerning

the Fourier transform of Kloosterman sums at Kurashiki and Aomori. Finally, I would like to express my gratitude to my family for always encouraging me.

This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

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Notation. Let p be an odd prime number. We fix a p -adic field F . We denote its ring of integers, its maximal ideal, and its residue field by \mathcal{O} , \mathfrak{p} , and k , respectively. We fix a uniformizer ϖ of F . Let q be the order of k . For $x \in \mathcal{O}$, \bar{x} denotes the image of x in k . We write Γ for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F .

For an abelian group T , we write $X^*(T)$ for its character group and $X_*(T)$ for its cocharacter group.

2. SIMPLE SUPERCUSPIDAL REPRESENTATIONS

2.1. Iwahori subgroups. Let G be a connected split reductive group over F , and Z its center. For simplicity, we often identify Z with the set of F -rational points $Z(F)$. Let T be an F -split maximal torus in G . We denote the set of roots of T in G by Φ , and the set of affine roots by Ψ . For each root $a \in \Phi$, we denote by U_a the corresponding root subgroup. For each affine root $\alpha \in \Psi$, we denote by U_α the corresponding affine root subgroup.

We fix an alcove C in the apartment $\mathcal{A}(G, T) \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ of T in $G(F)$. This determines an affine root basis Π of Ψ and the set Ψ^+ of positive affine roots. We set the Iwahori subgroup associated to C and its subgroups as follows:

$$\begin{aligned} I &:= \langle T_0, U_\alpha \mid \alpha \in \Psi^+ \rangle, \\ I^+ &:= \langle T_1, U_\alpha \mid \alpha \in \Psi^+ \rangle, \text{ and} \\ I^{++} &:= \langle T_1, U_\alpha \mid \alpha \in \Psi^+ \setminus \Pi \rangle, \end{aligned}$$

where T_0 is the maximal compact subgroup of $T(F)$, and

$$T_1 := \{t \in T_0 \mid \lambda(t) \in 1 + \mathfrak{p} \text{ for every } \lambda \in X^*(T)\}.$$

These groups are the first three steps of the Moy-Prasad filtration of the Iwahori subgroup I associated to the barycenter of the alcove C (see [RY14, Section 2.6]).

We define the subgroups of $T(F)$ by

$$\begin{aligned} T(q) &:= \{t \in T(F) \mid t^q = t\} \text{ and} \\ Z(q) &:= \{t \in Z(F) \mid t^q = t\}. \end{aligned}$$

These are sets of representatives of T_0/T_1 and $(Z \cap T_0)/(Z \cap T_1)$.

Proposition 2.1 ([GR10, Lemma 9.2]). (1) *The subgroup I^+ is normal in I , and we have*

$$I/I^+ \cong T_0/T_1 \cong T(q).$$

(2) *The subgroup I^{++} is normal in I^+ , and we have*

$$I^+/I^{++} \cong \bigoplus_{\alpha \in \Pi} U_\alpha/U_{\alpha+1}.$$

We denote the image of x under the map $I^+ \twoheadrightarrow \bigoplus_{\alpha \in \Pi} U_\alpha/U_{\alpha+1}$ by $(x_\alpha)_{\alpha \in \Pi}$, and call each x_α the *affine simple component* of x .

Definition 2.2. (1) An element $x \in I^+$ is said to be *affine generic* if x_α is nonzero for every $\alpha \in \Pi$.
(2) A character $\psi: I^+ \rightarrow \mathbb{C}^\times$ is called *affine generic* if it factors through the quotient I^+/I^{++} and is nontrivial on $U_\alpha/U_{\alpha+1}$ for every $\alpha \in \Pi$.

Proposition 2.3 ([HR08, Proposition 8 and Lemma 14]). *Let \widetilde{W} be the Iwahori-Weyl group of T defined by*

$$\widetilde{W} := N(F)/T_0,$$

where N is the normalizer of T in G . Then the following hold.

- (1) *We have $G(F) = IN(F)I$, and the map $InI \mapsto \dot{n}$ induces a bijection $I \backslash G(F)/I \cong \widetilde{W}$.*
- (2) *There exists an exact sequence*

$$1 \rightarrow W_{\text{aff}} \rightarrow \widetilde{W} \xrightarrow{\kappa_G} X^*(Z(\widehat{G})) \rightarrow 1,$$

where W_{aff} is the affine Weyl group of S , and κ_G is the Kottwitz homomorphism defined in [Kot97, Section 7]. Moreover the subgroup $\widetilde{\Omega} \subset \widetilde{W}$ consisting of the elements normalizing I maps isomorphically to $X^(Z(\widehat{G}))$, and we have $\widetilde{W} \cong W_{\text{aff}} \rtimes \widetilde{\Omega}$.*

By the same argument in the proof in [HR08], we can easily generalize this proposition as follows.

Proposition 2.4. *We have $G(F) = I^+ N(F) I^+$, and the map $I^+ n I^+ \mapsto \dot{n}$ induces a bijection*

$$I^+ \backslash G(F) / I^+ \cong N(F) / T_1.$$

We fix a set of representatives $\Omega \subset N(F)$ of $\tilde{\Omega} \subset \tilde{W} = N(F) / T_0$. Let $N_G(I)$ and $N_G(I^+)$ be the normalizers of I and I^+ in $G(F)$, respectively.

Lemma 2.5. *We have*

$$N_G(I) = N_G(I^+) = I\Omega.$$

Proof. Since I^+ is the pro-unipotent radical of I , we have $N_G(I) \subset N_G(I^+)$. We prove the other inclusion. Let $g \in N_G(I^+)$. It suffices to prove $I^+ g I^+ \subset N_G(I)$. By Proposition 2.4, we can replace g with $n \in N(F)$. As $n I^+ n^{-1} = I^+$ and $n T n^{-1} = T$, we have $n \in N_G(I)$. Hence $N_G(I) = N_G(I^+)$.

We next prove the second equality. The inclusion $N_G(I) \supset I\Omega$ follows from the definition of Ω . Let $g \in N_G(I)$. It suffices to prove $IgI \subset I\Omega$. By Proposition 2.3 (1), we may assume $g \in N(F)$. Then we have $g \in \Omega T_0$ by the definition of Ω . Hence $IgI \subset I\Omega$. \square

The following lemma is a key to compute characters.

Lemma 2.6. *Let $y \in G(F)$. If y satisfies $ygy^{-1} \in I$ for an affine generic element $g \in I^+$, then $y \in I\Omega$.*

Proof. Let $y \in G(F)$ satisfying $ygy^{-1} \in I$ for an affine generic element $g \in I^+$. Since affine genericity is preserved by I^+ -conjugation, any element of IyI^+ satisfies the same condition as y . Therefore, by Proposition 2.4, we may assume $y \in N(F)$.

We recall that the multiplication map

$$\prod_{a \in \Phi^+} U_a(F) \times T(F) \times \prod_{a \in \Phi^-} U_a(F) \rightarrow G(F)$$

is injective in any order (see [BT84, 2.2.3]). Moreover this induces a bijection

$$\prod_{a \in \Phi^+} U_{a+r_a} \times T_0 \times \prod_{a \in \Phi^-} U_{a+r_a} \rightarrow I$$

in any order, where $r_a \in \mathbb{Z}$ is the smallest integer such that $a + r_a \in \Psi^+$ (see [BT72, 6.4.9]). If we write

$$g = \prod_{a \in \Phi^+} x_a \cdot t \cdot \prod_{a \in \Phi^-} x_a,$$

where $t \in T_0$ and $x_a \in U_{a+r_a}$ for each $a \in \Phi$, then we have

$$ygy^{-1} = \prod_{a \in \Phi^+} yx_a y^{-1} \cdot yty^{-1} \cdot \prod_{a \in \Phi^-} yx_a y^{-1}.$$

Therefore, by the above uniqueness of expression, the assumption $ygy^{-1} \in I$ means that $yx_a y^{-1}$ belongs to I for every $a \in \Phi$. Since g is affine generic, this implies that $yU_\alpha y^{-1} \subset I$ for every affine simple root $\alpha \in \Pi$, hence for every positive affine root $\alpha \in \Psi^+$. On the other hand, we have $yT_0 y^{-1} = T_0$ since $y \in N(F)$. Thus $yIy^{-1} \subset I$.

Since g^{-1} is also affine generic, we have $yIy^{-1} \supset I$ by the same argument. Therefore $yIy^{-1} = I$ and y belongs to $N_G(I) = I\Omega$. \square

2.2. Simple supercuspidal representations. Let ψ be a character on ZI^+ such that $\psi|_{I^+}$ is affine generic. We put

$$N(\psi) := \{n \in N_G(I^+) \mid \psi^n = \psi\},$$

where ψ^n is a character of I^+ defined by $\psi^n(g) := \psi(ngn^{-1})$. This subgroup satisfies $ZI^+ \subset N(\psi) \subset I\Omega$ by Lemma 2.5. For an irreducible constituent χ of $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$, we define

$$\pi_\chi := \text{c-Ind}_{N(\psi)}^{G(F)} \chi.$$

Remark 2.7. (1) If G is semisimple and simply-connected, then the group $\tilde{\Omega}$ is trivial by Proposition 2.3 (2). Hence we have $N(\psi) = ZI^+$.

(2) In this paper, we will consider the cases of $G = \text{GL}_n$ and $G = \text{SO}_{2n+1}$. For these groups, the quotient $N(\psi) \twoheadrightarrow N(\psi)/ZI^+$ splits, and $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$ decomposes as a direct sum of characters.

Proposition 2.8 ([RY14, Proposition 2.4] and [GR10, Proposition 9.3]). (1)

We have a decomposition

$$\text{c-Ind}_{ZI^+}^{G(F)} \psi \cong \bigoplus_{\chi} \dim(\chi) \cdot \pi_\chi,$$

where the sum is over the set of irreducible constituents of $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$.

(2) *The representation π_χ is irreducible, hence supercuspidal.*

(3) *Let (ψ', χ') be another pair as above. Then, π_χ and $\pi_{\chi'}$ are equivalent if and only if $\psi^n = \psi'$ and $\chi^n \cong \chi'$ for some $n \in T_0\Omega$.*

Proof. Since ZI^+ is normal in $N(\psi)$ and the quotient $N(\psi)/ZI^+$ is finite, we have

$$\text{c-Ind}_{ZI^+}^{N(\psi)} \psi \cong \bigoplus_{\chi} \dim(\chi) \cdot \chi.$$

Hence it suffices to prove (2) and (3) by the transitivity of compact induction. By Mackey's theorem, we have

$$\text{Hom}_{G(F)}(\pi_\chi, \pi_{\chi'}) \cong \bigoplus_{n \in N(\psi) \backslash G(F)/N(\psi')} \text{Hom}_{N(\psi)^n \cap N(\psi')}(\chi^n, \chi').$$

Let $n \in G(F)$ such that $\text{Hom}_{N(\psi)^n \cap N(\psi')}(\chi^n, \chi') \neq 0$. Then we may assume $n \in N(F)$ by Proposition 2.4. Since $Z(I^+)^n \cap ZI^+ \subset N(\psi)^n \cap N(\psi)$, we also have $\text{Hom}_{Z(I^+)^n \cap ZI^+}(\chi^n, \chi') \neq 0$. As $\chi^n|_{Z(I^+)^n} = (\psi^n)^{\oplus \dim \chi}$, and $\chi'|_{ZI^+} = \psi'^{\oplus \dim \chi'}$, we have $\psi^n = \psi'$ on $Z(I^+)^n \cap ZI^+$.

We show that the image w of n under

$$I^+ \backslash G(F)/I^+ \twoheadrightarrow W_{\text{aff}} \rtimes \tilde{\Omega}$$

lies in $\tilde{\Omega}$. We assume that $w \notin \tilde{\Omega}$. Then we can take an affine simple root $\alpha \in \Pi$ such that $w(\alpha) \in \Psi^+ \setminus \Pi$ (see [GR10, Lemma 9.1]). By the definition of I^{++} , $U_{w(\alpha)}$ is contained in I^{++} . Hence ψ is trivial on $U_{w(\alpha)}$, and ψ^n is trivial on $n^{-1}U_{w(\alpha)}n = U_\alpha$. Since $U_\alpha = n^{-1}U_{w(\alpha)}n \subset Z(I^+)^n \cap ZI^+$, we have $\psi^n = \psi'$ on U_α . However ψ' is nontrivial on U_α since $\alpha \in \Pi$. This is a contradiction.

As $w \in \tilde{\Omega}$, n normalizes I , hence also normalizes I^+ . Therefore we have $ZI^+ \subset N(\psi)^n \cap N(\psi')$, and $\psi^n = \psi'$. Since $N(\psi)^n = N(\psi^n) = N(\psi')$, we have $\chi^n \cong \chi'$.

We finally show the irreducibility. We take $\chi = \chi'$. Then we have

$$\text{Hom}_{G(F)}(\pi_\chi, \pi_\chi) \cong \text{Hom}_{N(\psi)}(\chi, \chi),$$

and the dimension of this space is one. \square

The representations π_χ constructed in this way are called *simple supercuspidal* representations of $G(F)$.

2.3. Parametrization: the case of $\mathrm{GL}_N(F)$. In this subsection, we consider the case of GL_N . Let G be GL_N over F . We choose T to be the subgroup of diagonal matrices. Then we have

$$\begin{aligned}\Phi &= \{\pm(e_i - e_j) \mid 1 \leq i < j \leq N\}, \text{ and} \\ \Psi &= \{a + r \mid a \in \Phi, r \in \mathbb{Z}\}.\end{aligned}$$

We take the root basis

$$\Delta = \{e_1 - e_2, \dots, e_{N-1} - e_N\}$$

corresponding to the Borel subgroup B consisting of upper triangular matrices. We let C be the fundamental alcove of $\mathcal{A}(G, T)$ (i.e., C is contained in the chamber which is defined by B , and the closure \overline{C} of C contains 0). Then the corresponding affine root basis is

$$\Pi = \{e_1 - e_2, \dots, e_{N-1} - e_N, e_N - e_1 + 1\},$$

and the Iwahori subgroup and its filtrations are given by

$$\begin{aligned}I &= \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}, I^+ = \begin{pmatrix} 1 + \mathfrak{p} & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & 1 + \mathfrak{p} \end{pmatrix}, \text{ and} \\ I^{++} &= \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p} & & \mathcal{O} \\ & \ddots & \ddots & \\ & & \mathfrak{p} & \ddots \\ \mathfrak{p}^2 & & & 1 + \mathfrak{p} \end{pmatrix}.\end{aligned}$$

For $x = (x_{ij})_{ij} \in I^+$, we regard its affine simple components $(x_\alpha)_\alpha \in \bigoplus_{\alpha \in \Pi} U_\alpha / U_{\alpha+1}$ as an element of $k^{\oplus N}$ by

$$\begin{aligned}I^+ / I^{++} &\cong \bigoplus_{\alpha \in \Pi} U_\alpha / U_{\alpha+1} \cong k^{\oplus N} \\ (x_{ij})_{ij} &\mapsto (\overline{x_{12}}, \dots, \overline{x_{N-1,N}}, \overline{x_{N1}\varpi^{-1}}).\end{aligned}$$

For $a \in k^\times$, we set

$$\varphi_a := \begin{pmatrix} 0 & I_{N-1} \\ \varpi a & 0 \end{pmatrix} \in G(F).$$

Here, we regard a as an element of F^\times by the Teichmüller lift. This element satisfies $\varphi_a^N = \varpi a I_N$, and we can choose a set of representatives Ω of $\tilde{\Omega}$ to be $\langle \varphi_{a^{-1}} \rangle$.

We fix a nontrivial additive character $\psi: k \rightarrow \mathbb{C}^\times$. For $a \in k^\times$, we define a character $\psi_a: I^+ \rightarrow \mathbb{C}^\times$ by

$$\psi_a(x) := \psi(\overline{x_{12}} + \dots + \overline{x_{N-1,N}} + \overline{ax_{N1}\varpi^{-1}}) \text{ for } x = (x_{ij})_{ij} \in I^+.$$

Then we have $N(\psi_a) = ZI^+ \langle \varphi_{a^{-1}} \rangle$.

For an N -th root of unity $\zeta \in \mu_N$, let $\chi_{a,\zeta}: ZI^+\langle\varphi_{a^{-1}}\rangle \rightarrow \mathbb{C}^\times$ be the character defined by

$$\begin{aligned}\chi_{a,\zeta}(z) &= 1 \text{ for } z \in Z, \\ \chi_{a,\zeta}(x) &= \psi_a(x) \text{ for } x \in I^+, \text{ and} \\ \chi_{a,\zeta}(\varphi_{a^{-1}}) &= \zeta.\end{aligned}$$

Let $\pi_{a,\zeta}$ be the simple supercuspidal representation of $\mathrm{GL}_N(F)$ defined by

$$\pi_{a,\zeta} := \mathrm{c}\text{-Ind}_{ZI^+\langle\varphi_{a^{-1}}\rangle}^{G(F)} \chi_{a,\zeta}.$$

Then, by Proposition 2.8, we can check that the set

$$\{(a, \zeta) \mid a \in k^\times, \zeta \in \mu_N\}$$

parametrizes the set of equivalent classes of simple supercuspidal representations of $\mathrm{GL}_N(F)$ with trivial central characters.

2.4. Parametrization: the case of $\mathrm{SO}_{2n+1}(F)$. In this subsection, we consider the case of

$$\mathrm{SO}_{2n+1} := \{g \in \mathrm{GL}_{2n+1} \mid {}^t g J g = J, \det(g) = 1\},$$

with

$$J = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \ddots & & \\ & & & \ddots & \\ (-1)^{2n} & & & & \end{pmatrix}.$$

Let H be SO_{2n+1} over F . Let T_H be the subgroup of diagonal matrices in H :

$$T_H := \{\mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \neq 0\}.$$

Then we have

$$\begin{aligned}\Phi &= \{\pm e_i \pm e_j, \pm e_i \mid 1 \leq i < j \leq n\}, \text{ and} \\ \Psi &= \{a + r \mid a \in \Phi, r \in \mathbb{Z}\}.\end{aligned}$$

We take the root basis

$$\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$$

corresponding to the Borel subgroup B_H consisting of upper triangular matrices in H . We let C_H be the fundamental alcove of $\mathcal{A}(H, T_H)$. Then the corresponding affine root basis is

$$\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n, -e_1 - e_2 + 1\}.$$

We denote the Iwahori subgroup and its subgroups by I_H , I_H^+ , and I_H^{++} .

For $y = (y_{ij})_{ij} \in I_H^+$, we regard its affine simple components $(y_\alpha)_\alpha \in \bigoplus_{\alpha \in \Pi} U_\alpha / U_{\alpha+1}$ as an element of $k^{\oplus n+1}$ by

$$\begin{aligned}I^+ / I^{++} &\cong \bigoplus_{\alpha \in \Pi} U_\alpha / U_{\alpha+1} \cong k^{\oplus n+1} \\ (y_{ij})_{ij} &\mapsto (\overline{y_{12}}, \dots, \overline{y_{n,n+1}}, \overline{y_{2n,1} \varpi^{-1}}).\end{aligned}$$

For $b \in k^\times$, we set

$$\varphi'_b := - \begin{pmatrix} & \varpi^{-1}b^{-1} \\ I_{2n-1} & \\ \varpi b & \end{pmatrix} \in H(F).$$

Here, we regard b as an element of F^\times by the Teichmüller lift. The order of this element is two, and we can choose a set of representatives Ω of $\tilde{\Omega}$ to be $\langle \varphi'_{b^{-1}} \rangle$.

We fix a nontrivial additive character $\psi: k \rightarrow \mathbb{C}^\times$. For $b \in k^\times$, we define a character $\psi'_b: I_H^+ \rightarrow \mathbb{C}^\times$ by

$$\psi'_b(y) := \psi \left(\overline{y_{12}} + \cdots + \overline{y_{n,n+1}} + \overline{by_{2n,1}\varpi^{-1}} \right) \text{ for } y = (y_{ij})_{ij} \in I_H^+.$$

Then we have $N(\psi'_b) = I_H^+ \langle \varphi'_{b^{-1}} \rangle$.

For $\xi \in \{\pm 1\}$, let $\chi'_{b,\xi}: I_H^+ \langle \varphi'_{b^{-1}} \rangle \rightarrow \mathbb{C}^\times$ be the character defined by

$$\begin{aligned} \chi'_{b,\xi}(y) &= \psi'_b(y) \text{ for } y \in I_H^+ \text{ and} \\ \chi'_{b,\xi}(\varphi'_{b^{-1}}) &= \xi. \end{aligned}$$

Let $\pi'_{b,\xi}$ be the simple supercuspidal representation of $\mathrm{SO}_{2n+1}(F)$ defined by

$$\pi'_{b,\xi} := \mathrm{c}\text{-Ind}_{I_H^+ \langle \varphi'_{b^{-1}} \rangle}^{H(F)} \chi'_{b,\xi}.$$

Then, by Proposition 2.8, we can check that the set

$$\{(b, \xi) \mid b \in k^\times, \xi \in \{\pm 1\}\}$$

parametrizes the set of equivalent classes of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$.

3. CHARACTERS OF SIMPLE SUPERCUSPIDAL REPRESENTATIONS

3.1. The case of standard $\mathrm{GL}_N(F)$. Let us first recall the characters of representations of a p -adic reductive group. For a connected reductive group G over F , we write $G^{\mathrm{rs}}(F)$ for the set of regular semisimple elements of $G(F)$. This is an open subset of $G(F)$. We denote by $\mathcal{H}(G)$ the set of compactly supported locally constant functions on $G(F)$.

Theorem 3.1 ([HC70]). *Let G be a connected reductive group over F . Let π be an irreducible smooth representation of $G(F)$. Then there exists a unique locally constant function Θ_π on $G^{\mathrm{rs}}(F)$ such that*

$$\mathrm{tr} \pi(f) = \int_{G^{\mathrm{rs}}(F)} \Theta_\pi(g) f(g) dg$$

for every $f \in \mathcal{H}(G)$ satisfying $\mathrm{supp}(f) \subset G^{\mathrm{rs}}(F)$, where $\mathrm{tr} \pi$ is the distribution character of π .

We call Θ_π the *character* of π . This function is invariant under conjugation.

When π is a supercuspidal representation which is compactly induced from an open subgroup, we have the following formula to describe its character.

Theorem 3.2 (Character formula, [Sal88]). *Let G be a connected reductive group over F and Z its center. Let K be an open subgroup of $G(F)$ such that K contains Z and K/Z is compact. Let ρ be a finite-dimensional irreducible smooth representation*

of K . We assume that the representation $\pi := \text{c-Ind}_K^{G(F)} \rho$ is supercuspidal. Then we have

$$\Theta_\pi(g) = \sum_{\substack{y \in K \backslash G(F) \\ ygy^{-1} \in K}} \text{tr}(\rho(ygy^{-1}))$$

for every $g \in G^{\text{rs}}(F)$, where Θ_π is the character of π .

We apply this formula to a computation of the characters of simple supercuspidal representations.

Let G be GL_N over F , and K the subgroup $ZI^+\langle\varphi_1\rangle$ of $G(F)$. We take $\zeta \in \mu_N$. Let π be the simple supercuspidal representation $\pi_{1,\zeta}$ of $G(F)$. We denote $\chi_{1,\zeta}$ and φ_1 simply by χ and φ , respectively.

Lemma 3.3. *Let $g \in I^+ \subset \text{GL}_N(F)$ be an affine generic element. Then g is regular semisimple elliptic.*

Proof. The characteristic polynomial of $g - I_N$ is Eisenstein, hence it is irreducible over F . Therefore g is regular semisimple elliptic. \square

Proposition 3.4. *Let $g \in I^+$ be an affine generic element. Let (g_1, \dots, g_N) be the affine simple components of g . Then we have*

$$\Theta_\pi(g) = \text{Kl}_{g_1 \dots g_N}^N(\psi),$$

where the right-hand side is the Kloosterman sum in Definition A.3.

We first prove the following lemma.

Lemma 3.5. *Let $g \in I^+$ be an affine generic element. If $y \in G(F)$ satisfies $ygy^{-1} \in ZI^+\langle\varphi\rangle$, then $y \in ZI\langle\varphi\rangle$.*

Proof. Assume that $y \in G(F)$ satisfies $ygy^{-1} \in ZI^+\langle\varphi\rangle$. As $\det(ygy^{-1}) = \det(g)$, ygy^{-1} lies in I . Thus $y \in I\Omega = ZI\langle\varphi\rangle$ by Lemma 2.6. \square

Proof of Proposition 3.4. By Lemma 3.5, if $y \in G(F)$ satisfies $ygy^{-1} \in ZI^+\langle\varphi\rangle$, then $y \in ZI\langle\varphi\rangle$. Hence, by the character formula (Theorem 3.2), we have

$$\Theta_\pi(g) = \sum_{y \in ZI^+\langle\varphi\rangle \backslash ZI\langle\varphi\rangle} \chi(ygy^{-1}).$$

Since

$$\{t = \text{diag}(t_1, \dots, t_N) \in T(q) \mid t_N = 1\}$$

is a system of representatives of the set $ZI^+\langle\varphi\rangle \backslash ZI\langle\varphi\rangle$, we have

$$\begin{aligned} \text{RHS} &= \sum_{t_1, \dots, t_{N-1} \in k^\times} \psi\left(\frac{t_1}{t_2}g_1 + \dots + \frac{t_{N-1}}{1}g_{N-1} + \frac{1}{t_1}g_N\right) \\ &= \sum_{\substack{s_1, \dots, s_N \in k^\times \\ s_1 \dots s_N = g_1 \dots g_N}} \psi(s_1 + \dots + s_N) \\ &= \text{Kl}_{g_1 \dots g_N}^N(\psi). \end{aligned}$$

\square

3.2. The case of twisted $\mathrm{GL}_{2n}(F)$. In this subsection, we compute the twisted characters of self-dual simple supercuspidal representations of $\mathrm{GL}_{2n}(F)$ with trivial central characters.

Let us first recall the twisted characters of representations of a p -adic reductive group. For a connected reductive group G over F and its automorphism θ over F , we write $G^{\theta\text{-rs}}(F)$ for the set of θ -regular θ -semisimple elements of $G(F)$ (see 1.1 and 3.3 in [KS99] for the definitions of θ -semisimplicity and θ -regularity). This is an open subset of $G(F)$.

Theorem 3.6 ([Lem10, 5.8 Corollaire]). *Let G be a connected reductive group over F . Let θ be an automorphism of G defined over F . Let π be a θ -stable (i.e., $\pi \cong \pi^\theta$) irreducible smooth representation of $G(F)$, and fix an isomorphism $A: \pi \rightarrow \pi^\theta$. Then there exists a unique locally constant function $\Theta_{\pi,\theta}$ on $G^{\theta\text{-rs}}(F)$ such that*

$$\mathrm{tr} \pi_\theta(f) = \int_{G^{\theta\text{-rs}}(F)} \Theta_{\pi,\theta}(g) f(g) dg$$

for every $f \in \mathcal{H}(G)$ satisfying $\mathrm{supp}(f) \subset G^{\theta\text{-rs}}(F)$, where $\mathrm{tr} \pi_\theta$ is the θ -twisted distribution character of π with respect to A .

We call $\Theta_{\pi,\theta}$ the θ -twisted character of π . This function is invariant under θ -conjugation, and depends on an isomorphism $A: \pi \cong \pi^\theta$ (determined up to scalar multiple).

Similarly in the standard case, we have a formula for the twisted character of supercuspidal representations.

Theorem 3.7 (Twisted character formula, [Lem10, 6.2 Théorème]). *Let G be a reductive group over F and Z its center. Let θ be an automorphism of G over F . Let K be a θ -stable open subgroup of $G(F)$ such that K contains Z and K/Z is compact. Let ρ be a finite-dimensional θ -stable irreducible smooth representation of K . We fix an isomorphism $A: \rho \rightarrow \rho^\theta$. We assume that the representation $\pi := \mathrm{c}\text{-Ind}_K^{G(F)} \rho$ is supercuspidal. Then we have*

$$\Theta_{\pi,\theta}(g) = \sum_{\substack{y \in K \setminus G(F) \\ yg\theta(y)^{-1} \in K}} \mathrm{tr}(\rho(yg\theta(y)^{-1}) \circ A)$$

for $g \in G^{\theta\text{-rs}}(F)$, where $\Theta_{\pi,\theta}$ is the θ -twisted character of π with respect to the isomorphism $\mathrm{c}\text{-Ind}_K^G A: \pi \rightarrow \pi^\theta$.

Let G be GL_{2n} over F , and K_a the subgroup $ZI^+\langle\varphi_{a-1}\rangle$ of $G(F)$ for $a \in k^\times$. Let θ be the automorphism of G over F defined by

$$\theta(g) = J^t g^{-1} J^{-1}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & \ddots & \\ (-1)^{2n-1} & & & \end{pmatrix}.$$

For $g \in G(F)$, we put $N(g) := g\theta(g) \in G(F)$.

The automorphism θ preserves the subgroups I^+ and I^{++} , and induces the automorphism of $I^+/I^{++} \cong k^{\oplus 2n}$ defined by

$$(x_1, \dots, x_{2n-1}, x_{2n}) \mapsto (x_{2n-1}, \dots, x_1, x_{2n}).$$

In particular we have $\psi_a^\theta = \psi_a$ for $a \in k^\times$. On the other hand, by a simple computation, we can check $\theta(\varphi_{a^{-1}}) = -\varphi_{a^{-1}}^{-1}$. Therefore

$$\pi_{a,\zeta}^\theta \cong \text{c-Ind}_{K_a}^{G(F)} \chi_{a,\zeta}^\theta \cong \text{c-Ind}_{K_a}^{G(F)} \chi_{a,\zeta^{-1}} = \pi_{a,\zeta^{-1}},$$

and $\pi_{a,\zeta}$ is self-dual if and only if $\zeta = \pm 1$.

Let $\zeta \in \{\pm 1\}$, and we put $\pi := \pi_{1,\zeta}$. We denote K_1 , $\chi_{1,\zeta}$, and φ_1 simply by K , χ , and φ , respectively. We fix an automorphism $A := \text{id}$ of χ . Then this defines the twisted character $\Theta_{\pi,\theta}$ of π .

First, we compute the twisted character $\Theta_{\pi,\theta}$ at $g \in I^+ \cap G^{\theta\text{-rs}}(F)$ such that $N(g) = g\theta(g) \in I^+$ is affine generic.

Lemma 3.8. *Let $g \in I^+$ be an element such that $N(g)$ is affine generic. If $y \in G(F)$ satisfies $yg\theta(y)^{-1} \in ZI^+\langle\varphi\rangle$, then $y \in ZI\langle\varphi\rangle$.*

Proof. Since $yg\theta(y)^{-1} \in K$, $N(yg\theta(y)^{-1}) = yN(g)y^{-1}$ belongs to $K\theta(K) = K$. By the assumption and Lemma 3.5, y must lie in $ZI\langle\varphi\rangle$. \square

Lemma 3.9. *Let $g \in I^+$ be an element such that $N(g)$ is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+\langle\varphi\rangle \setminus ZI\langle\varphi\rangle \mid yg\theta(y)^{-1} \in K\}$$

is given by

$$T'(q) := \{\text{diag}(t_1, \dots, t_{2n}) \in T(q) \mid t_1 t_{2n} = \dots = t_n t_{n+1}, t_n = 1\}.$$

Proof. Let $y := \text{diag}(t_1, \dots, t_{2n}) \in T(q)$ satisfying $t_n = 1$. Since

$$\text{val}(\det(yg\theta(y)^{-1})) = 0,$$

we have

$$yg\theta(y)^{-1} \in K \Rightarrow yg\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have $yg\theta(y)^{-1} \in K$ if and only if the diagonal part of

$$\begin{aligned} yg\theta(y)^{-1} &= \text{diag}(t_1, \dots, t_{2n}) \begin{pmatrix} g_{1,1} & \dots & g_{1,2n} \\ \vdots & \ddots & \vdots \\ g_{2n,1} & \dots & g_{2n,2n} \end{pmatrix} \text{diag}(t_{2n}, \dots, t_1) \\ &= \begin{pmatrix} t_1 t_{2n} g_{1,1} & t_1 t_{2n-1} g_{1,2} & & \\ & t_2 t_{2n-1} g_{2,2} & \ddots & * \\ & & \ddots & t_{2n-1} t_1 g_{2n-1,2n} \\ t_{2n}^2 g_{2n,1} & * & & t_{2n} t_1 g_{2n,2n} \end{pmatrix} \end{aligned}$$

lies in $Z(q)T_1$. Therefore the condition $yg\theta(y)^{-1} \in K$ is equivalent to that $t_1 t_{2n} = \dots = t_n t_{n+1}$. \square

Proposition 3.10. *Let $g \in I^+ \cap G^{\theta\text{-rs}}(F)$ be an element such that $N(g)$ is affine generic. Let (g_1, \dots, g_{2n}) be the affine simple components of g . Then we have*

$$\Theta_{\pi,\theta}(g) = \text{Kl}_{g_n(g_1+g_{2n-1})^2 \dots (g_{n-1}+g_{n+1})^2 g_{2n}}^{n+1}(\psi; 1, 2, \dots, 2, 1),$$

where the right-hand side is the Kloosterman sum in Definition A.3.

Proof. Note that the affine simple components of $N(g)$ are given by

$$(g_1 + g_{2n-1}, \dots, g_{2n-1} + g_1, 2g_{2n}),$$

and the affine genericity of $N(g)$ means that none of them is zero.

By the twisted character formula (Theorem 3.7) and Lemma 3.9, we can compute the twisted character as follows:

$$\begin{aligned} \Theta_{\pi, \theta}(g) &= \sum_{y \in T'(q)} \chi(yg\theta(y)^{-1}) \\ &= \sum_{a \in k^\times} \sum_{\substack{t_1, \dots, t_n \in k^\times \\ t_i t_{2n+1-i} = a, t_n = 1}} \psi \left(\frac{t_1 t_{2n-1} g_1}{a} + \dots + \frac{t_{2n-1} t_1 g_{2n-1}}{a} + \frac{t_{2n}^2 g_{2n}}{a} \right) \\ &= \sum_{a \in k^\times} \sum_{t_1, \dots, t_{n-1} \in k^\times} \psi \left(\frac{t_1}{t_2} (g_1 + g_{2n-1}) + \dots + \frac{t_{n-1}}{1} (g_{n-1} + g_{n+1}) + \frac{1}{a} g_n + \frac{a}{t_1^2} g_{2n} \right) \\ &= \sum_{t_1, \dots, t_n \in k^\times} \psi \left(\frac{t_1}{t_2} (g_1 + g_{2n-1}) + \dots + \frac{t_{n-1}}{1} (g_{n-1} + g_{n+1}) + \frac{t_n}{t_1} g_n + \frac{1}{t_1 t_n} g_{2n} \right) \\ &= \sum_{t_1, \dots, t_n \in k^\times} \psi \left(\frac{t_1}{t_2} g_n + \frac{t_2}{t_3} (g_1 + g_{2n-1}) + \dots + \frac{t_n}{1} (g_{n-1} + g_{n+1}) + \frac{1}{t_1 t_2} g_{2n} \right) \\ &= \text{Kl}_{g_n(g_1+g_{2n-1})^2 \dots (g_{n-1}+g_{n+1})^2 g_{2n}}^{n+1}(\psi; 1, 2, \dots, 2, 1). \end{aligned}$$

Here,

- 4th equality: we replaced a with t_1/t_n ,
- 5th equality: we permutated indices of t by $(1 \ 2 \ \dots \ n)$.

□

Next, we compute the twisted character $\Theta_{\pi, \theta}$ at $\varphi_u g$, where $g \in I^+$ and $u \in k^\times$ such that $-N(\varphi_u g) = \varphi_u g \varphi_u^{-1} \theta(g) \in I^+$ is affine generic.

Lemma 3.11. *Let $g \in I^+$ be an element such that $-N(\varphi_u g) \in I^+$ is affine generic. If $y \in G(F)$ satisfies $y\varphi_u g\theta(y)^{-1} \in ZI^+\langle\varphi\rangle$, then $y \in ZI\langle\varphi\rangle$.*

Proof. Since $y\varphi_u g\theta(y)^{-1} \in K$, $N(y\varphi_u g\theta(y)^{-1}) = yN(\varphi_u g)y^{-1}$ belongs to $K\theta(K) = K$. By the assumption and Lemma 3.5, y must lie in $ZI\langle\varphi\rangle$. □

Lemma 3.12. *Let $g \in I^+$ be an element such that $-N(\varphi_u g) \in I^+$ is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+\langle\varphi\rangle \setminus ZI\langle\varphi\rangle \mid y\varphi_u g\theta(y)^{-1} \in K\}$$

is given by

$$T''(q) := \{\text{diag}(t_1, \dots, t_{2n}) \in T(q) \mid t_1 t_{2n-1} = \dots = t_{2n-1} t_1 = u, t_{2n} = 1\}.$$

Proof. Let $y = \text{diag}(t_1, \dots, t_{2n}) \in T(q)$ satisfying $t_{2n} = 1$. Since

$$\text{val}(\det(y\varphi_u g\theta(y)^{-1})) = \text{val}(\det(\varphi)),$$

we have

$$y\varphi_u g\theta(y)^{-1} \in K = ZI^+\langle\varphi\rangle \Rightarrow \varphi^{-1}y\varphi_u g\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have $y\varphi_u g\theta(y)^{-1} \in K$ if and only if the diagonal part of

$$\varphi^{-1}y\varphi_u \cdot g \cdot \theta(y)^{-1}$$

$$\begin{aligned}
&= \text{diag}(t_{2n}u, t_1, \dots, t_{2n-1}) \begin{pmatrix} g_{1,1} & \cdots & g_{1,2n} \\ \vdots & \ddots & \vdots \\ g_{2n,1} & \cdots & g_{2n,2n} \end{pmatrix} \text{diag}(t_{2n}, t_{2n-1}, \dots, t_1) \\
&= \begin{pmatrix} t_{2n}^2 u g_{1,1} & t_{2n} t_{2n-1} u g_{1,2} & & \\ & t_1 t_{2n-1} g_{2,2} & \ddots & * \\ & * & \ddots & t_{2n-2} t_1 g_{2n-1,2n} \\ t_{2n-1} t_{2n} g_{2n,1} & & & t_{2n-1} t_1 g_{2n,2n} \end{pmatrix}
\end{aligned}$$

lies in $Z(q)T_1$. Therefore that $y\varphi_u g\theta(y)^{-1} \in K$ is equivalent to that $t_1 t_{2n-1} = \dots = t_{2n-1} t_1 = u$. \square

Proposition 3.13. *Let $g \in I^+$ be an element such that $\varphi_u g \in G^{\theta\text{-rs}}(F)$ and $-N(\varphi_u g)$ is affine generic. Let (g_1, \dots, g_{2n}) be the affine simple components of g .*

(1) *If $u \notin k^{\times 2}$, then we have*

$$\Theta_{\pi, \theta}(\varphi_u g) = 0.$$

(2) *If $u = v^2$ for some $v \in k^\times$, then we have*

$$\Theta_{\pi, \theta}(\varphi_u g) = \chi(\varphi) \left(\text{Kl}_{(v^2 g_1 + g_{2n}) \cdots (g_n + g_{n+1})/v}^n(\psi) + \text{Kl}_{-(v^2 g_1 + g_{2n}) \cdots (g_n + g_{n+1})/v}^n(\psi) \right).$$

Proof. Note that the affine simple components of $-N(\varphi_u g)$ are given by

$$(g_2 + g_{2n-1}, g_3 + g_{2n-2}, \dots, g_{2n-1} + g_2, u^{-1} g_{2n} + g_1, u g_1 + g_{2n}),$$

and the affine genericity of $N(g)$ means that none of them is zero.

We use the twisted character formula (Theorem 3.7) and Lemma 3.12. If $u \notin k^{\times 2}$, then the set $T''(q)$ is empty. Hence the sum in the twisted character formula is zero.

If $u = v^2$ for some $v \in k^\times$, then we can compute the twisted character as follows:

$$\begin{aligned}
\Theta_{\pi, \theta}(\varphi_{v^2} g) &= \sum_{y \in T''(q)} \chi(\varphi) \chi(\varphi^{-1} y \varphi_{v^2} g \theta(y)^{-1}) \\
&= \chi(\varphi) \sum_{\substack{t_1, \dots, t_{2n} \in k^\times \\ t_i t_{2n-i} = v^2 \\ t_{2n} = 1}} \psi \left(\frac{t_{2n} t_{2n-1} v^2 g_1}{v^2} + \frac{t_1 t_{2n-2} g_2}{v^2} + \cdots + \frac{t_{2n-2} t_1 g_{2n-1}}{v^2} + \frac{t_{2n-1} t_{2n} g_{2n}}{v^2} \right) \\
&= \chi(\varphi) \sum_{\substack{t_1, \dots, t_n \in k^\times \\ t_n = \pm v}} \psi \left(\frac{1}{t_1} (v^2 g_1 + g_{2n}) + \frac{t_1}{t_2} (g_2 + g_{2n-1}) + \cdots + \frac{t_{n-1}}{t_n} (g_n + g_{n+1}) \right) \\
&= \chi(\varphi) \left(\text{Kl}_{(v^2 g_1 + g_{2n}) \cdots (g_n + g_{n+1})/v}^n(\psi) + \text{Kl}_{-(v^2 g_1 + g_{2n}) \cdots (g_n + g_{n+1})/v}^n(\psi) \right).
\end{aligned}$$

\square

3.3. The case of $\text{SO}_{2n+1}(F)$. Let H be SO_{2n+1} . We take $b \in k^\times$ and $\xi \in \{\pm 1\}$. Let K' be the subgroup $I_H^+(\varphi'_{b^{-1}})$ of $H(F)$. We put $\pi' := \pi'_{b, \xi}$. We denote $\chi'_{b, \xi}$ and $\varphi'_{b^{-1}}$ simply by χ' and φ' , respectively.

First, we compute the character of π' at an affine generic element $h \in I_H^+ \cap H^{\text{rs}}(F)$.

Lemma 3.14. *Let $h \in I_H^+$ be an affine generic element. If $y \in H(F)$ satisfies $y h y^{-1} \in I_H^+(\varphi')$, then $y \in I_H(\varphi')$.*

Proof. If $yhy^{-1} \in I_H^+(\varphi')$, then $(yhy^{-1})^2 = yh^2y^{-1} \in I_H^+$. Since we assumed that the characteristic of k is not equal to 2, h^2 is also affine generic. Therefore $y \in I_H\langle\varphi'\rangle$ by Lemma 2.6. \square

Proposition 3.15. *Let $h \in I_H^+ \cap H^{\text{rs}}(F)$ be an affine generic element with its affine simple components $(h_1, \dots, h_n, h_{2n})$. Then we have*

$$\Theta_{\pi'}(h) = \text{Kl}_{h_1h_2^2 \dots h_n^2 h_{2n}b}^{n+1}(\psi; 1, 2, \dots, 2, 1).$$

Proof. By the character formula (Theorem 3.2) and Lemma 3.14, we can compute the character as follows:

$$\begin{aligned} \Theta_{\pi'}(h) &= \sum_{y \in I_H^+(\varphi') \setminus I_H\langle\varphi'\rangle} \chi'(yhy^{-1}) = \sum_{t \in T_H(q)} \chi'(tht^{-1}) \\ &= \sum_{t_1, \dots, t_n \in k^\times} \psi \left(\frac{t_1}{t_2} h_1 + \dots + \frac{t_{n-1}}{t_n} h_{n-1} + \frac{t_n}{1} h_n + \frac{b}{t_1 t_2} h_{2n} \right) \\ &= \sum_{\substack{s_1, \dots, s_n, s_{2n} \in k^\times \\ s_1 s_2^2 \dots s_n^2 s_{2n} = h_1 h_2^2 \dots h_n^2 h_{2n} b}} \psi(s_1 + \dots + s_n + s_{2n}) \\ &= \text{Kl}_{h_1 h_2^2 \dots h_n^2 h_{2n} b}^{n+1}(\psi; 1, 2, \dots, 2, 1). \end{aligned}$$

\square

Remark 3.16. Later, we will prove that every affine generic element of $H(F)$ is strongly regular semisimple (Proposition 4.5).

Next, we compute the character of π' at $\varphi'_{b^{-1}u}h$, where $h \in I_H^+$ and $u \in k^\times$ such that $(\varphi'_{b^{-1}u}h)^2 \in I_H^+$ is affine generic.

Lemma 3.17. *Let $h \in I_H^+$ be an element such that $(\varphi'_{b^{-1}u}h)^2$ is affine generic. If $y \in H(F)$ satisfies $y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')$, then $y \in I_H\langle\varphi'\rangle$.*

Proof. If $y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')$, then $(y\varphi'_{b^{-1}u}hy^{-1})^2 = y(\varphi'_{b^{-1}u}h)^2y^{-1} \in I_H^+$. By the assumption and Lemma 2.6, y belongs to $I_H\langle\varphi'\rangle$. \square

Lemma 3.18. *Let $h \in I_H^+$ be an element such that $(\varphi'_{b^{-1}u}h)^2$ is affine generic. Then the set*

$$\{y \in I_H^+(\varphi') \setminus I_H\langle\varphi'\rangle \mid y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')\}$$

is represented by

$$T'_H(q) := \{ \text{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \in T_H(q) \mid t_1^2 = u \}.$$

Proof. Let $y = \text{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \in T_H(q)$ be an element such that $y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')$. We have a decomposition $I_H^+(\varphi') = I_H^+ \amalg \varphi' I_H^+$, and we can check $y\varphi'_{b^{-1}u}hy^{-1} \notin I_H^+$, hence $y\varphi'_{b^{-1}u}hy^{-1}$ belongs to $\varphi' I_H^+$. Thus we have $y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')$ if and only if the diagonal part of

$$\varphi'^{-1} y \varphi'_{b^{-1}u} \cdot h \cdot y^{-1}$$

$$\begin{aligned}
&= \text{diag}(t_1^{-1}u, t_2, t_3, \dots) \begin{pmatrix} h_{1,1} & \dots & h_{1,2n+1} \\ \vdots & \ddots & \vdots \\ h_{2n+1,1} & \dots & h_{2n+1,2n+1} \end{pmatrix} \text{diag}(t_1^{-1}, t_2^{-1}, t_3^{-1}, \dots) \\
&= \begin{pmatrix} t_1^{-2}uh_{1,1} & t_1^{-1}t_2^{-1}uh_{1,2} & & \\ & h_{2,2} & \ddots & * \\ & & \ddots & t_1t_2^{-1}h_{2n,2n+1} \\ t_1^{-1}t_2^{-1}h_{2n,1} & * & & t_1^2u^{-1}h_{2n+1,2n+1} \\ & t_1t_2^{-1}u^{-1}h_{2n+1,2} & & \end{pmatrix}
\end{aligned}$$

lies in $T_H(1 + \mathfrak{p})$. Therefore $y\varphi'_{b^{-1}u}hy^{-1} \in I_H^+(\varphi')$ is equivalent to $t_1^2 = u$. \square

Proposition 3.19. *Let $h \in I_H^+$ be an element such that $\varphi'_{b^{-1}u}h \in H^{\text{rs}}(F)$ and $(\varphi'_{b^{-1}u}h)^2$ is affine generic. Let $(h_1, \dots, h_n, h_{2n})$ be the affine simple components of h .*

(1) *If $u \notin k^{\times 2}$, then we have*

$$\Theta_{\pi'}(\varphi'_{b^{-1}u}h) = 0.$$

(2) *If $u = v^2$ for some $v \in k^\times$, then we have*

$$\Theta_{\pi'}(\varphi'_{b^{-1}u}h) = \chi'(\varphi') \left(\text{Kl}_{(h_1v^2+h_{2n}b)h_2 \dots h_n/v}^n(\psi) + \text{Kl}_{-(h_1v^2+h_{2n}b)h_2 \dots h_n/v}^n(\psi) \right).$$

Proof. Note that the affine simple components of $(\varphi'_{b^{-1}u}h)^2$ are given by

$$(h_{2n}bu^{-1} + h_1, 2h_2, \dots, 2h_n, h_1b^{-1}u + h_{2n}),$$

and the affine genericity of $N(g)$ means that none of them is zero.

We use the character formula (Theorem 3.2) and Lemma 3.18. If $u \notin k^{\times 2}$, then the set $T'_H(q)$ is empty. Hence the sum in the character formula is zero.

If $u = v^2$ for some $v \in k^\times$, then we can compute the character as follows:

$$\begin{aligned}
\Theta_{\pi'}(\varphi'_{b^{-1}v^2}h) &= \sum_{y \in T'_H(q)} \chi'(\varphi') \chi'(\varphi'^{-1}y\varphi'_{b^{-1}v^2}hy^{-1}) \\
&= \chi'(\varphi') \sum_{\substack{t_1, \dots, t_n \in k^\times \\ t_1 = \pm v}} \psi \left(\frac{1}{t_1t_2}h_1v^2 + \frac{t_2}{t_3}h_2 + \dots + \frac{t_n}{1}h_n + \frac{b}{t_1t_2}h_{2n} \right) \\
&= \chi'(\varphi') \left(\text{Kl}_{(h_1v^2+h_{2n}b)h_2 \dots h_n/v}^n(\psi) + \text{Kl}_{-(h_1v^2+h_{2n}b)h_2 \dots h_n/v}^n(\psi) \right).
\end{aligned}$$

\square

4. NORM CORRESPONDENCES

4.1. Norm correspondences. Let us first recall the norm correspondence for twisted endoscopy. Our basic reference in this section is [KS99].

Let G be a connected quasi-split reductive group over F , and θ an automorphism of G defined over F . Let $(H, {}^LH, s, \xi)$ be endoscopic data for the triplet $(G, \theta, 1)$. Then we have the norm map

$$\mathcal{A}_{H/G}: Cl_{\text{ss}}(H) \rightarrow Cl_{\theta\text{-ss}}(G, \theta)$$

from the set of semisimple conjugacy classes in $H(\overline{F})$ to the set of θ -semisimple θ -conjugacy classes in $G(\overline{F})$ (see Section 3.3 in [KS99]).

Let $x \in G(F)$ be a strongly θ -regular θ -semisimple element and $y \in H(F)$ a strongly regular semisimple element. We say that y is a *norm* of x if x corresponds to y via the map $\mathcal{A}_{H/G}$.

In this section, we consider the following situation:

- $G = \mathrm{GL}_{2n}$ over F ,
- $\theta(g) = J^t g^{-1} J^{-1}$, where

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & & \\ & & \ddots & \\ (-1)^{2n-1} & & & \end{pmatrix},$$

- $H = \mathrm{SO}_{2n+1}$ over F ,
- $s = 1$, and
- $\xi: {}^L\mathrm{SO}_{2n+1} = \mathrm{Sp}_{2n}(\mathbb{C}) \times W_F \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_F = {}^L\mathrm{GL}_{2n}$.

For $g \in G(\overline{F})$, we put $N(g) := g\theta(g) \in G(\overline{F})$. In this section, we let T_0 and $T_{H,0}$ be the subgroups of diagonal matrices in G and H , respectively. Then we can write the map $\mathcal{A}_{H/G}$ explicitly in terms of T_0 and $T_{H,0}$ as follows:

$$\begin{aligned} Cl_{ss}(H) &\xleftarrow{\cong} T_{H,0}(\overline{F})/\Omega_{T_{H,0}} \xrightarrow{\cong} (T_0)_\theta(\overline{F})/\Omega_{T_0}^\theta \xrightarrow{\cong} Cl_{\theta-ss}(G, \theta) \xleftarrow{\quad} T_0(\overline{F}) \\ &\quad \mathrm{diag}\left(\frac{t_1}{t_{2n}}, \dots, \frac{t_n}{t_{n+1}}, 1, \frac{t_{n+1}}{t_n}, \dots, \frac{t_{2n}}{t_1}\right) \longleftrightarrow \mathrm{diag}(t_1, \dots, t_{2n}), \end{aligned}$$

where $\Omega_{T_{H,0}}$ is the Weyl group of $T_{H,0}$ in H , $\Omega_{T_0}^\theta$ is the θ -fixed part of the Weyl group of T_0 in G , and $(T_0)_\theta$ is the θ -coinvariant part of T_0 . Note that the map $\mathcal{A}_{H/G}$ is an isomorphism since $T_{H,0}(\overline{F}) \cong (T_0)_\theta(\overline{F})$ and $\Omega_{T_{H,0}} \cong \Omega_{T_0}^\theta$ in our situation.

Let us prove some lemmas needed later.

Lemma 4.1. *Let x be a θ -semisimple element in $G(\overline{F})$ and y a semisimple element in $H(\overline{F})$ which corresponds to x via $\mathcal{A}_{H/G}$. If $N(x) = x\theta(x) \in G(\overline{F})$ is regular, then x is strongly θ -regular and y is strongly regular.*

Proof. Since $\mathrm{Cent}_G(x, \theta) \subset \mathrm{Cent}_G(N(x))$, x is a strongly θ -regular (recall the definition of strongly θ -regular elements). By the assumption that y corresponds to x , y is also strongly regular ([KS99, Lemma 3.3.C]). \square

Lemma 4.2. *Let x be a θ -semisimple element in $G(\overline{F})$ and y a semisimple element in $H(\overline{F})$. If y is conjugate to a matrix*

$$\begin{pmatrix} 1 & * \\ 0 & N(x) \end{pmatrix}$$

in $\mathrm{GL}_{2n+1}(\overline{F})$, then x corresponds to y via $\mathcal{A}_{H/G}$.

Proof. Let $t = \mathrm{diag}(t_1, \dots, t_{2n}) \in T_0(\overline{F})$ be an element which is θ -conjugate to x in $G(\overline{F})$. Then, by the assumption, y is conjugate to

$$t_H := \mathrm{diag}\left(\frac{t_1}{t_{2n}}, \dots, \frac{t_n}{t_{n+1}}, 1, \frac{t_{n+1}}{t_n}, \dots, \frac{t_{2n}}{t_1}\right) \in T_{H,0}(\overline{F})$$

in $\mathrm{GL}_{2n+1}(\overline{F})$.

If semisimple elements $y_1, y_2 \in \mathrm{SO}_{2n+1}(\overline{F})$ are conjugate in $\mathrm{GL}_{2n+1}(\overline{F})$, then they are conjugate in $\mathrm{SO}_{2n+1}(\overline{F})$ (see [SS70, 4.2]). Therefore y is conjugate to t_H in $\mathrm{SO}_{2n+1}(\overline{F})$, and x corresponds to y via $\mathcal{A}_{H/G}$. \square

Lemma 4.3. *Let h be a strongly regular semisimple elliptic element in $H(F)$. Then there exists a strongly θ -regular θ -semisimple θ -elliptic element $g \in G(F)$ such that h is a norm of g .*

Proof. This follows from the adjoint relation of the transfer factor for strongly regular semisimple elliptic elements (see the proof of [Art13, Proposition 2.1.1]). \square

4.2. Norm correspondences for affine generic elements. In this subsection, we study the norm correspondence for affine generic elements. We use the same notations as in Subsections 2.3 and 2.4.

Lemma 4.4. *Let $h \in I_H^+ \subset H(F)$ be an affine generic element with its affine simple components $(h_1, \dots, h_n, h_{2n})$. Then h is conjugate to a matrix*

$$\begin{pmatrix} 1 & * \\ 0 & h' \end{pmatrix}$$

in $\mathrm{GL}_{2n+1}(F)$, where h' is an affine generic element of $\mathrm{GL}_{2n}(F)$ with its affine simple components $(h_2, \dots, h_n, h_n, \dots, h_1, 2h_{2n})$.

Proof. Let

$$h := \begin{pmatrix} h_{1,1} & \dots & h_{1,2n+1} \\ \vdots & \ddots & \vdots \\ h_{2n+1,1} & \dots & h_{2n+1,2n+1} \end{pmatrix} \in I_H^+$$

be an affine generic element with its affine simple components $(h_1, \dots, h_n, h_{2n})$, then we have

$$\begin{aligned} h_{1,2} &\equiv h_{2n,2n+1} \equiv h_1 \neq 0, \\ &\vdots \\ h_{n,n+1} &\equiv h_{n+1,n+2} \equiv h_n \neq 0, \text{ and} \\ h_{2n,1}\varpi^{-1} &\equiv h_{2n+1,2}\varpi^{-1} \equiv h_{2n} \neq 0 \pmod{\mathfrak{p}}. \end{aligned}$$

Since h is an element of $\mathrm{SO}_{2n+1}(F)$, h has an eigenvector with eigenvalue 1. We take such an eigenvector satisfying

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_{2n+1} \end{pmatrix} \in \mathcal{O}^{\oplus 2n+1} \setminus \mathfrak{p}^{\oplus 2n+1}.$$

Then we have

$$(*) \quad \begin{pmatrix} h_{1,1} - 1 & \dots & h_{1,2n+1} \\ \vdots & \ddots & \vdots \\ h_{2n+1,1} & \dots & h_{2n+1,2n+1} - 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_{2n+1} \end{pmatrix} = 0.$$

In particular, this gives

$$\begin{pmatrix} 0 & h_{1,2} & \dots & h_{1,2n+1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_{2n,2n+1} \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_{2n+1} \end{pmatrix} \equiv 0 \pmod{\mathfrak{p}}.$$

Hence we have

$$v_2 \equiv \cdots \equiv v_{2n+1} \equiv 0 \pmod{\mathfrak{p}}.$$

As $v \notin \mathfrak{p}^{\oplus 2n+1}$, we may assume that $v_1 = 1$.

Consider the following matrix:

$$\begin{aligned} & \begin{pmatrix} 1 & & & \\ & v_2 & \ddots & 0 \\ & \vdots & & \ddots \\ v_{2n+1} & 0 & & 1 \end{pmatrix}^{-1} h \begin{pmatrix} 1 & & & \\ & v_2 & \ddots & 0 \\ & \vdots & & \ddots \\ v_{2n+1} & 0 & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & h_{1,2} & \cdots & h_{1,2n+1} \\ 0 & h_{2,2} - v_2 h_{1,2} & \cdots & h_{2,2n+1} - v_2 h_{1,2n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{2n+1,2} - v_{2n+1} h_{1,2} & \cdots & h_{2n+1,2n+1} - v_{2n+1} h_{1,2n+1} \end{pmatrix}. \end{aligned}$$

Then it suffices to show that the matrix

$$h' := \begin{pmatrix} h_{2,2} - v_2 h_{1,2} & \cdots & h_{2,2n+1} - v_2 h_{1,2n+1} \\ \vdots & \ddots & \vdots \\ h_{2n+1,2} - v_{2n+1} h_{1,2} & \cdots & h_{2n+1,2n+1} - v_{2n+1} h_{1,2n+1} \end{pmatrix}$$

is a desired affine generic element in $\mathrm{GL}_{2n}(F)$. As h' is an element of I^+ , our task is to compute the affine simple components of h' .

First, as

$$v_2 \equiv \cdots \equiv v_{2n+1} \equiv 0 \pmod{\mathfrak{p}},$$

the first $(2n-1)$ -components of h' are $(h_2, \dots, h_n, h_n, \dots, h_1)$.

Second, by the $(2n)$ -th row of the equation $(*)$, we have

$$h_{2n,1} + h_{2n,2}v_2 + \cdots + (h_{2n,2n} - 1)v_{2n} + h_{2n,2n+1}v_{2n+1} = 0.$$

Thus we have

$$h_{2n,1} + h_{2n,2n+1}v_{2n+1} \equiv 0 \pmod{\mathfrak{p}^2},$$

so the last component of h' is

$$\begin{aligned} (h_{2n+1,2} - v_{2n+1}h_{1,2})\varpi^{-1} &\equiv h_{2n} - v_{2n+1}\varpi^{-1}h_1 \\ &\equiv 2h_{2n} \pmod{\mathfrak{p}}. \end{aligned}$$

□

Proposition 4.5. *Let $h \in I_H^+ \subset H(F)$ be an affine generic element with its affine simple components $(h_1, \dots, h_n, h_{2n})$. Then h is strongly regular semisimple elliptic, and there exists $g \in G(F)$ satisfying the following conditions:*

- g is strongly θ -regular θ -semisimple θ -elliptic,
- h is a norm of g , and
- $N(g)$ is an affine generic element of $G(F)$ with affine simple components $(h_2, \dots, h_n, h_n, \dots, h_1, 2h_{2n})$.

Proof. From Lemmas 4.4 and 3.3, it follows that h is semisimple. Since the centralizer of h in $H(F)$ is compact by Lemma 3.14, h is elliptic.

Let h' be the affine generic element of $G(F)$ defined in Lemma 4.4. We take a θ -semisimple element $x \in G(\overline{F})$ corresponding to h via $\mathcal{A}_{H/G}$, then $N(x)$ and h' are conjugate in $G(\overline{F})$. By θ -conjugation, we can replace x with $x' \in G(\overline{F})$ satisfying

$N(x') = h'$. By Lemma 3.3, $N(x') = h'$ is a strongly regular element of $G(F)$. Therefore h is a strongly regular element of $H(F)$ by Lemma 4.1.

Finally, we take a strongly θ -regular θ -semisimple θ -elliptic element $g' \in G(F)$ such that h is a norm of g' . Such an element exists by Lemma 4.3. Then h' and $N(g')$ are conjugate in $G(\overline{F})$, therefore also in $G(F)$. Hence we can replace g' with $g \in G(F)$ satisfying $N(g) = h'$ by θ -conjugation. This element is a desired one. \square

4.3. Transfer factors. As in the previous subsection, $G = \mathrm{GL}_{2n}$ and $H = \mathrm{SO}_{2n+1}$. Let $G^{\theta\text{-srs}}(F)$ be the set of strongly θ -regular θ -semisimple elements in $G(F)$, and $H^{\text{srs}}(F)$ the set of strongly regular semisimple elements in $H(F)$.

We fix the following θ -stable Whittaker datum (B_0, λ) of G :

- B_0 is the subgroup of upper triangular matrices in G , and
- λ is the character of the unipotent radical $N_0(F)$ of $B_0(F)$ defined by

$$\lambda(x) = \psi_F(x_{12} + \cdots + x_{2n-1, 2n}) \text{ for } x = (x_{ij}) \in N_0(F),$$

where ψ_F is a fixed nontrivial additive character of F .

Then we have the normalized absolute transfer factor $\Delta_{H,G}$ for G and H . This is a function

$$\Delta_{H,G}: H^{\text{srs}}(F) \times G^{\theta\text{-srs}}(F) \rightarrow \mathbb{C},$$

which has the following properties.

- The value $\Delta_{H,G}(\gamma, \delta)$ is nonzero only if γ is a norm of δ .
- If $\gamma_1, \gamma_2 \in H^{\text{srs}}(F)$ are stably conjugate, then $\Delta_{H,G}(\gamma_1, \delta) = \Delta_{H,G}(\gamma_2, \delta)$.
- If $\delta_1, \delta_2 \in G^{\theta\text{-srs}}(F)$ are θ -conjugate, then $\Delta_{H,G}(\gamma, \delta_1) = \Delta_{H,G}(\gamma, \delta_2)$.

Our purpose in this subsection is to compute $\Delta_{H,G}$ for affine generic elements. The transfer factor $\Delta_{H,G}$ is defined as the product of Δ_{I} , Δ_{II} , Δ_{III} and Δ_{IV} (since G and H are split, the contribution of the local ε -factor is trivial). To define these four factors, we have to fix several auxiliary data. We explain a part of them (see Sections 4 and 5 in [KS99] for the details).

Let $\hat{\theta}$ be an automorphism of \widehat{G} which is dual to θ . We first fix

- a $(\Gamma, \hat{\theta})$ -stable splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of G and
- a Γ -stable splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ of H

such that $s \in \mathcal{T}$ and $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$.

We take $\delta \in G^{\theta\text{-srs}}(F)$, and let $\gamma \in H^{\text{srs}}(F)$ be a norm of δ . For these elements, we fix

- a θ -stable pair (B, T) of G and
- a pair (B_H, T_H) of H , where $T_H := \mathrm{Cent}_H(\gamma)$

such that the isomorphism

$$T_H \cong T_{\theta}$$

which is induced by pairs (B, T) , $(\mathcal{B}, \mathcal{T})$, (B_H, T_H) , and $(\mathcal{B}_H, \mathcal{T}_H)$ is defined over F (we can take such pairs, see [KS99, Lemma 3.3.B]). Then, by the definition of the norm correspondence, there exist $g \in G_{\mathrm{sc}}(\overline{F})$ and $\delta^* \in T(\overline{F})$ satisfying the following conditions:

- γ has the image δ^* under $T_H \cong T_{\theta}$ and
- $\delta^* = g\delta\theta(g)^{-1}$.

We write $R(G, T)$ for the set of roots of T in G . Let R_{res} denote the set of restricted roots of $R(G, T)$:

$$R_{\text{res}} := \{ \alpha_{\text{res}} := \alpha|_{(T^\theta)^0} \mid \alpha \in R(G, T) \},$$

and R_{res}^\vee those of $R(\widehat{G}, \mathcal{T})$. We fix a -data and χ -data for R_{res} . By the bijection $\alpha_{\text{res}} \mapsto (\alpha^\vee)_{\text{res}}$ from R_{res} to R_{res}^\vee , we transport the Γ -action, a -data, and χ -data of R_{res} to R_{res}^\vee . Here, we identify R_{res}^\vee with the set of restricted coroots of T in G :

$$\{ (\alpha^\vee)_{\text{res}} := \alpha^\vee|_{(\widehat{T}^\theta)_0} \mid \alpha^\vee \in R^\vee(G, T) \},$$

by the canonical isomorphism $R^\vee(G, T) \cong R(\widehat{G}, \mathcal{T})$.

Lemma 4.6. *Let $\gamma \in H^{\text{srs}}(F)$ and $\delta \in G^{\theta\text{-srs}}(F)$. If γ is a norm of δ , then we have*

$$\Delta_{\text{I}}(\gamma, \delta) = 1.$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the Tate-Nakayama pairing between $H^1(F, T_{\text{sc}}^\theta)$ and $\pi_0(\widehat{T_{\text{sc}}^\theta}^\Gamma)$. Let $s_{T, \theta}$ be the image of s under the projection

$$\mathcal{T} \cong \widehat{T} \twoheadrightarrow (\widehat{T}_{\text{ad}})_\theta \cong \widehat{T_{\text{sc}}^\theta}.$$

In fact $s_{T, \theta}$ is invariant under Γ ([KS99, Lemma 4.2]). We denote the image of $s_{T, \theta}$ under the map

$$\widehat{T_{\text{sc}}^\theta}^\Gamma \twoheadrightarrow \pi_0(\widehat{T_{\text{sc}}^\theta}^\Gamma)$$

by \mathbf{s} . Then $\Delta_{\text{I}}(\gamma, \delta)$ is defined to be $\langle \lambda, \mathbf{s} \rangle$, where λ is a 1-cocycle determined by the fixed Whittaker data and a -data (see Section 4.2 in [KS99]). Since $s = 1$ in our setting, $\Delta_{\text{I}}(\gamma, \delta)$ is trivial. \square

Lemma 4.7. *Let $\gamma \in H^{\text{srs}}(F)$ and $\delta \in G^{\theta\text{-srs}}(F)$. If γ is a norm of δ , then we have*

$$\Delta_{\text{II}}(\gamma, \delta) = 1.$$

Proof. The set of restricted roots R_{res} is reduced. Moreover the set of restricted coroots R_{res}^\vee coincides with the set $R^\vee(H, T_H)$ of coroots of T_H in H . Therefore $\Delta_{\text{II}}(\gamma, \delta)$ is trivial by Lemma 4.3.A in [KS99]. \square

We next recall the definition of Δ_{III} . Let $\langle \cdot, \cdot \rangle$ be the Tate-Nakayama pairing

$$H^1(F, T_{\text{sc}} \rightarrow T) \times H^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}}) \rightarrow \mathbb{C},$$

where the maps $T_{\text{sc}} \rightarrow T$ and $\widehat{T} \rightarrow \widehat{T}_{\text{ad}}$ are $1 - \theta$ and $1 - \hat{\theta}$, respectively. Then the third factor $\Delta_{\text{III}}(\gamma, \delta)$ is defined to be $\langle \text{inv}(\gamma, \delta), \mathbf{A}_0 \rangle$, where $\text{inv}(\gamma, \delta)$ and \mathbf{A}_0 are 1-hypercycles defined as follows:

- Let $v_0: \Gamma \rightarrow T_{\text{sc}}(\overline{F})$ be a map defined by $v_0(\sigma) = g\sigma(g)^{-1}$. In fact this is a 1-cocycle and defines a 1-hypercycle

$$(v_0^{-1}, \delta^*) \in Z^1(F, T_{\text{sc}} \rightarrow T).$$

We denote by $\text{inv}(\gamma, \delta)$ the image of (v_0^{-1}, δ^*) in $H^1(F, T_{\text{sc}} \rightarrow T)$.

- Let $\widehat{G}^1 := (\widehat{G}^\theta)^0$, ${}^L G^1 := \widehat{G}^1 \rtimes W_F$, and $\mathcal{T}^1 := \mathcal{T} \cap \widehat{G}^1$. Since $R(\widehat{G}^1, \mathcal{T}^1) \subset R_{\text{res}}^\vee$, the Γ -action and χ -data for $R(\widehat{G}^1, \mathcal{T}^1)$ are induced from those of R_{res}^\vee . These data define an L -embedding ${}^L(T_\theta) \hookrightarrow {}^L G^1$ (see Section 2.6 in [LS87]). By composing the fixed admissible embedding ${}^L(T_H) \cong {}^L(T_\theta)$ and a natural embedding ${}^L G^1 \hookrightarrow {}^L G$, we get

$$\xi_1: {}^L(T_H) \cong {}^L(T_\theta) \hookrightarrow {}^L G^1 \hookrightarrow {}^L G.$$

On the other hand, since $s \in \mathcal{T}$, we also have $\mathcal{T}^1 \subset \xi(\widehat{H})$ and $R(\xi(\widehat{H}), \mathcal{T}^1) \subset R_{\text{res}}^\vee$. Similarly as above, we get an L -embedding ${}^L(T_H) \hookrightarrow {}^LH$. By composing ξ , we get

$$\xi_2: {}^L(T_H) \hookrightarrow {}^LH \xrightarrow{\xi} {}^LG.$$

Then these two L -embeddings define a 1-cocycle $a_T: W_F \rightarrow \mathcal{T}$ such that $\xi_2 = \xi_1 \cdot a_T$. We regard a_T as a 1-cocycle to \widehat{T} via $\mathcal{T} \cong \widehat{T}$. In fact this 1-cocycle defines a 1-hypercocycle

$$(a_T^{-1}, s) \in Z^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}}).$$

We denote by \mathbf{A}_0 the image of (a_T^{-1}, s) in $H^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}})$.

Lemma 4.8. *Let $\gamma \in H^{\text{srs}}(F)$ and $\delta \in G^{\theta\text{-srs}}(F)$. If γ is a norm of δ , then we have*

$$\Delta_{\text{III}}(\gamma, \delta) = 1.$$

Proof. By the definition of the third factor, it is enough to show that \mathbf{A}_0 is trivial. In our setting, s is trivial. Hence $\xi(H)$ is exactly equal to \widehat{G}^1 , and we have $a_T(\sigma) = \xi(1 \rtimes \sigma)$ for every $\sigma \in \Gamma$. However, since G and H are split and ξ is a natural L -embedding, the 1-cocycle a_T is trivial, and so is \mathbf{A}_0 . \square

Finally, we consider the fourth factor. Let D_H be the Weyl discriminant:

$$D_H(\gamma) := |\det(\text{Ad}(\gamma) - 1 \mid \mathfrak{h}/\mathfrak{t}_H)|^{\frac{1}{2}} \text{ for } \gamma \in H^{\text{srs}}(F),$$

where \mathfrak{h} and \mathfrak{t}_H are the Lie algebras of H and T_H , respectively. Let $D_{G,\theta}$ be the twisted Weyl discriminant:

$$D_{G,\theta}(\delta) := |\det(\text{Ad}(\delta) \circ \theta - 1 \mid \mathfrak{g}/\mathfrak{t})|^{\frac{1}{2}} \text{ for } \delta \in G^{\theta\text{-srs}}(F),$$

where \mathfrak{g} and \mathfrak{t} are the Lie algebras of G and $\text{Cent}_G(\text{Cent}_G(\delta, \theta)^0)$, respectively. The fourth factor is defined by

$$\Delta_{\text{IV}}(\gamma, \delta) = D_{G,\theta}(\delta)/D_H(\gamma).$$

Lemma 4.9. *Let $\gamma \in H^{\text{srs}}(F)$ and $\delta \in G^{\theta\text{-srs}}(F)$. If γ is a norm of δ , then we have*

$$\Delta_{\text{IV}}(\gamma, \delta) = 1$$

(note that this is equivalent to $D_{G,\theta}(\delta) = D_H(\gamma)$).

Proof. By [KS99, Lemma 4.5.A] and the same argument in the proof of Lemma 4.7, we get the result. \square

Proposition 4.10. *Let $\gamma \in H^{\text{srs}}(F)$ and $\delta \in G^{\theta\text{-srs}}(F)$. If γ is a norm of δ , then the transfer factor $\Delta_{H,G}(\gamma, \delta)$ is equal to 1.*

Proof. We have

$$\Delta_{H,G}(\gamma, \delta) = \Delta_{\text{I}}(\gamma, \delta) \Delta_{\text{II}}(\gamma, \delta) \Delta_{\text{III}}(\gamma, \delta) \Delta_{\text{IV}}(\gamma, \delta),$$

and this equals 1 by Lemmas 4.6, 4.7, 4.8 and 4.9. \square

4.4. Norms of $1+\varphi_u$ and $\varphi_u(1+\varphi_u)$. We first prove a lemma about θ -semisimplicity of semisimple elements in $G(F)$.

Lemma 4.11. *Let $g \in G(\overline{F})$ be a semisimple element such that $N(g) = g\theta(g)$ is regular semisimple. If $g\theta(g) = \theta(g)g$, then g is θ -semisimple.*

Proof. Since F has characteristic zero, g is θ -semisimple if and only if $\text{Int}(g) \circ \theta$ preserves a pair (B_G, T_G) in G (see Section 1.1 in [KS99]). If there exists $x \in G(\overline{F})$ such that $xg\theta(x)^{-1} \in T_0(\overline{F})$, then the pair $(x^{-1}B_0x, x^{-1}T_0x)$ is $\text{Int}(g) \circ \theta$ -invariant. Therefore it suffices to prove the existence of such $x \in G(\overline{F})$.

As g is semisimple, $\theta(g)$ is also semisimple. Since $g\theta(g) = \theta(g)g$, we can take $x \in G(\overline{F})$ such that $xgx^{-1} \in T_0(\overline{F})$ and $x\theta(g)x^{-1} \in T_0(\overline{F})$. Note that this element also satisfies $xN(g)x^{-1} \in T_0(\overline{F})$. Since the element $N(g)$ is θ -invariant, we can replace x so that $xN(g)x^{-1} \in T_0^\theta(\overline{F})$ by Ω_{T_0} -conjugation.

Then we have

$$xN(g)x^{-1} = \theta(xN(g)x^{-1}) = \theta(x)N(g)\theta(x)^{-1},$$

hence $x^{-1}\theta(x)$ belongs to $\text{Cent}_G(N(g))$. By the regularity of $N(g)$, $\text{Cent}_G(N(g)) = x^{-1}T_0x$. Therefore $x\theta(x)^{-1} \in T_0(\overline{F})$, and $xg\theta(x)^{-1} = xgx^{-1} \cdot x\theta(x)^{-1} \in T_0(\overline{F})$. This element x is a desired one. \square

For $u \in k^\times$, we consider

$$1 + \varphi_u = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ \varpi u & & & 1 \end{pmatrix} \in I^+ \subset \text{GL}_{2n}(F), \text{ and}$$

$$N'(1 + \varphi_u) := (1 - \varpi u)^{-1} \begin{pmatrix} 1 & & & & \\ \varpi u & 1 + \varpi u & & & 2 \\ \vdots & & \ddots & & \\ \varpi u & 2\varpi u & & 1 + \varpi u & \\ (\varpi u)^2/2 & \varpi u & \dots & \varpi u & 1 \end{pmatrix}$$

$$\in I_H^+ \subset \text{SO}_{2n+1}(F).$$

These are affine generic elements.

Proposition 4.12. *The element $1 + \varphi_u \in G(F)$ is strongly θ -regular θ -semisimple and $N'(1 + \varphi_u) \in H(F)$ is strongly regular semisimple. Moreover, $N'(1 + \varphi_u)$ is a norm of $1 + \varphi_u$.*

Proof. We first show that $1 + \varphi_u$ is θ -semisimple. By Lemma 3.3, $1 + \varphi_u$ is semisimple. Since

$$\begin{aligned} \theta(1 + \varphi_u) &= J(1 + {}^t\varphi_u)^{-1}J^{-1} \\ &= J(1 - {}^t\varphi_u + {}^t\varphi_u^2 - \dots)J^{-1} \\ &= 1 + \varphi_u + \varphi_u^2 + \dots, \end{aligned}$$

$1 + \varphi_u$ commutes with $\theta(1 + \varphi_u)$. Moreover we can check that

$$N(1 + \varphi_u) = (1 + \varphi_u)\theta(1 + \varphi_u) = (1 - \varpi u)^{-1} \begin{pmatrix} 1 + \varpi u & & 2 \\ & \ddots & \\ 2\varpi u & & 1 + \varpi u \end{pmatrix}.$$

Since the characteristic of k is not equal to 2, this is an affine generic element of $G(F)$. Hence this is regular semisimple by Lemma 3.3. Therefore $1 + \varphi_u$ is θ -semisimple by Lemma 4.11.

We next show $N'(1 + \varphi_u)$ is semisimple and corresponds to $1 + \varphi_u$ via $\mathcal{A}_{H/G}$. Let X be the matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ -\varpi u/2 & & 1 \end{pmatrix},$$

then we have

$$X^{-1}N'(1 + \varphi_u)X = (1 - \varpi u)^{-1} \begin{pmatrix} 1 - \varpi u & & 2 \\ 0 & 1 + \varpi u & \\ \vdots & & \ddots \\ 0 & 2\varpi u & & 1 + \varpi u \end{pmatrix}.$$

By the affine genericity of $N(1 + \varphi_u)$, this element is semisimple. Moreover $N'(1 + \varphi_u)$ corresponds to $1 + \varphi_u$ via $\mathcal{A}_{H/G}$ by Lemma 4.2.

Finally, by Lemma 4.1, $1 + \varphi_u$ is strongly θ -regular and $N'(1 + \varphi_u)$ is strongly regular. \square

Next, for $u \in k^\times$ we consider the elements

$$\begin{aligned} \varphi_u(1 + \varphi_u) &\in \mathrm{GL}_{2n}(F), \text{ and} \\ \varphi'_{u/2}N'(1 + \varphi_u) &= (-1 + \varpi u)^{-1} \begin{pmatrix} \varpi u & & & 2/(\varpi u) \\ \varpi u & 1 + \varpi u & & 2 \\ \vdots & & \ddots & \\ \varpi u & 2\varpi u & & 1 + \varpi u \\ \varpi u/2 & \varpi u & \dots & \varpi u & \varpi u \end{pmatrix} \\ &\in \mathrm{SO}_{2n+1}(F). \end{aligned}$$

Proposition 4.13. *The element $\varphi_u(1 + \varphi_u) \in G(F)$ is strongly θ -regular θ -semisimple and $\varphi'_{u/2}N'(1 + \varphi_u) \in H(F)$ is strongly regular semisimple. Moreover, $\varphi'_{u/2}N'(1 + \varphi_u)$ is a norm of $\varphi_u(1 + \varphi_u)$.*

Proof. We first show that $\varphi_u(1 + \varphi_u)$ is θ -semisimple. Since the characteristic polynomial of φ_u is $t^{2n} - \varpi u$ and irreducible over F , φ_u is semisimple. Hence so is $\varphi_u(1 + \varphi_u)$. Since

$$\begin{aligned} \theta(\varphi_u(1 + \varphi_u)) &= -\varphi_u^{-1}J(1 - {}^t\varphi_u + {}^t\varphi_u^2 - \dots)J^{-1} \\ &= -\varphi_u^{-1}(1 + \varphi_u + \varphi_u^2 + \dots), \end{aligned}$$

$\varphi_u(1 + \varphi_u)$ commutes with $\theta(\varphi_u(1 + \varphi_u))$. Moreover we have

$$\begin{aligned} N(\varphi_u(1 + \varphi_u)) &= \varphi_u(1 + \varphi_u)\theta(\varphi_u(1 + \varphi_u)) = -N(1 + \varphi_u) \\ &= -(1 - \varpi u)^{-1} \begin{pmatrix} 1 + \varpi u & & 2 \\ & \ddots & \\ 2\varpi u & & 1 + \varpi u \end{pmatrix}. \end{aligned}$$

Since the characteristic of k is not equal to 2, $-N(\varphi_u(1 + \varphi_u))$ is an affine generic element of $G(F)$. Hence this is regular semisimple by Lemma 3.3. Therefore $\varphi_u(1 + \varphi_u)$ is θ -semisimple by Lemma 4.11.

We next show $\varphi'_{u/2}N'(1 + \varphi_u)$ is semisimple and corresponds to $1 + \varphi_u$ via $\mathcal{A}_{H/G}$. Let X be the matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ -\varpi u/2 & & 1 \end{pmatrix},$$

then we have

$$X^{-1}\varphi'_{u/2}N'(1 + \varphi_u)X = (-1 + \varpi u)^{-1} \begin{pmatrix} -1 + \varpi u & 2 & \dots & 2/(\varpi u) \\ 0 & 1 + \varpi u & & 2 \\ \vdots & & \ddots & \\ 0 & 2\varpi u & & 1 + \varpi u \end{pmatrix}.$$

By the affine genericity of $-N(\varphi_u(1 + \varphi_u))$, this element is semisimple. Moreover $\varphi'_{u/2}N'(1 + \varphi_u)$ corresponds to $\varphi_u(1 + \varphi_u)$ via $\mathcal{A}_{H/G}$ by Lemma 4.2.

Finally, by Lemma 4.1, $\varphi_u(1 + \varphi_u)$ is strongly θ -regular and $\varphi'_{u/2}N'(1 + \varphi_u)$ is strongly regular. \square

5. MAIN THEOREM

5.1. Endoscopic character relation. We first recall the endoscopic classification of representations of classical groups in [Art13].

Let G be GL_N over F . Let θ be the automorphism of G over F defined by

$$g \mapsto J^t g^{-1} J^{-1}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{N-1} & & & \end{pmatrix}.$$

Let H be a simple endoscopic group of the triplet $(G, \theta, 1)$. This is either a symplectic group or an orthogonal group. We denote the group of outer automorphisms of the endoscopic group H by $\widetilde{\mathrm{Out}}_N(H)$. This group is nontrivial only if H is an even orthogonal group.

We write $\widetilde{\Phi}(H)$ for the set of self-dual L -parameters of G which factor through ${}^L H$. We denote by ϕ_H the L -parameter of H defined by $\phi \in \widetilde{\Phi}(H)$. For $\phi \in \widetilde{\Phi}(H)$, we set

$$\begin{aligned} S_\phi &:= \mathrm{Cent}_{\widehat{H}}(\mathrm{Im}(\phi_H)), \text{ and} \\ \mathcal{S}_\phi &:= S_\phi / S_\phi^0 Z(\widehat{H})^\Gamma. \end{aligned}$$

Theorem 5.1 ([Art13, Theorems 1.5.1 and 2.2.1]). *For every bounded self-dual L -parameter $\phi \in \tilde{\Phi}(H)$, there is a finite set $\tilde{\Pi}_\phi$ consisting of $\tilde{\text{Out}}_N(H)$ -orbits of irreducible tempered representations of $H(F)$, and the following properties hold.*

- *There is a bijection from $\tilde{\Pi}_\phi$ to the group $\widehat{\mathcal{S}}_\phi$ of characters of \mathcal{S}_ϕ .*
- *For every $f \in \mathcal{H}(G)$, we have the following equality of stable distributions:*

$$\text{tr } \pi_\theta(f) = \sum_{\pi_H \in \tilde{\Pi}_\phi} \text{tr } \pi_H(f^H),$$

where π is the representation of $G(F)$ corresponding to ϕ via the local Langlands correspondence for GL_N , $\text{tr } \pi_\theta$ is its θ -twisted distribution character with respect to the normalization determined by the Whittaker datum in Section 4.3, and $f^H \in \mathcal{H}(H)$ is a transfer of f to H .

From now on, we consider the case $N = 2n$.

Proposition 5.2 ([Mie15]). *Let π be a self-dual simple supercuspidal representation of $G(F)$ with trivial central character. Then the L -parameter of π belongs to $\tilde{\Phi}(\text{SO}_{2n+1})$.*

By Theorem 5.1 and Proposition 5.2, a self-dual simple supercuspidal representation π of G defines a finite set $\tilde{\Pi}_\phi$ consisting of irreducible representations of $H(F) = \text{SO}_{2n+1}(F)$. Since π is supercuspidal, the corresponding L -parameter $\phi: W_F \rightarrow \widehat{G} = \text{GL}_{2n}(\mathbb{C})$ is irreducible as a representation of W_F . Therefore $\text{Cent}_{\widehat{G}}(\text{Im}(\phi))$ consists of scalar matrices, and so does $\text{Cent}_{\widehat{H}}(\text{Im}(\phi_H))$. Hence the group \mathcal{S}_ϕ is trivial and $\tilde{\Pi}_\phi$ is a singleton by Theorem 5.1. We denote by π_H the unique representation in $\tilde{\Pi}_\phi$ and say that π_H is *associated to* π . We remark that the character Θ_{π_H} of π_H is stable by Theorem 5.1.

Our purpose is to determine π_H by using the relation in Theorem 5.1.

$$\begin{array}{ccc} \pi & \xrightarrow{\text{LLC for } \text{GL}_{2n}} & W_F \xrightarrow{\phi} L_G \\ & \searrow \text{Theorem 5.1} & \searrow \phi_H \downarrow \xi \\ \tilde{\Pi}_\phi = \{\pi_H\} & & L_H \end{array}$$

By using a stable version of the twisted Weyl integration formula, we can rewrite the relation in Theorem 5.1 in terms of the characters $\Theta_{\pi, \theta}$ and Θ_{π_H} as follows.

Theorem 5.3. *Let π be a self-dual simple supercuspidal representation of $G(F)$ with trivial central character. Let π_H be the irreducible representation of $H(F)$ associated to π . Let $\Theta_{\pi, \theta}$ be the twisted character of π with respect to the normalization in Section 3.2, and Θ_{π_H} the character of π_H . Let $c \in \mathbb{C}^\times$ be the ratio of the Whittaker normalization to the normalization in Section 3.2. Then, for every $g \in G^{\theta\text{-srs}}(F)$, we have the following equality:*

$$c \cdot \Theta_{\pi, \theta}(g) = \sum_{h \mapsto g} \frac{D_H(h)^2}{D_{G, \theta}(g)^2} \Delta_{H, G}(h, g) \Theta_{\pi_H}(h),$$

where the sum is over stable conjugacy classes of norms $h \in H^{\text{srs}}(F)$ of g .

We can simplify this equality by the bijectivity of $\mathcal{A}_{H/G}$ and the computation of the transfer factor $\Delta_{H, G}$ in Section 4.3.

Lemma 5.4. *Let $g \in G^{\theta\text{-srs}}(F)$. Then g has at most one norm in $H^{\text{srs}}(F)$ up to stable conjugacy.*

Proof. Let $h, h' \in H^{\text{srs}}(F)$ be norms of g . In our situation, the norm map $\mathcal{A}_{H/G}$ is bijective, hence h and h' are conjugate in $\text{SO}_{2n+1}(\overline{F})$. Since h and h' are strongly regular, they are stably conjugate. \square

Corollary 5.5. *Let $g \in G^{\theta\text{-srs}}(F)$ and $h \in H^{\text{srs}}(F)$ such that h is a norm of g . Then we have*

$$c \cdot \Theta_{\pi, \theta}(g) = \Theta_{\pi_H}(h).$$

Proof. Combining Theorem 5.3 with Lemma 5.4, we have

$$c \cdot \Theta_{\pi, \theta}(g) = \frac{D_H(h)^2}{D_{G, \theta}(g)^2} \Delta_{H, G}(h, g) \Theta_{\pi_H}(h).$$

Moreover, by Proposition 4.10 and Lemma 4.9, we have

$$c \cdot \Theta_{\pi, \theta}(g) = \Theta_{\pi_H}(h).$$

\square

Corollary 5.6. *Let $\zeta \in \{\pm 1\}$. Let $\pi := \pi_{1, \zeta}$ and π_H its associated representation. Then, for $u \in k^\times$, we have*

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = c \cdot \Theta_{\pi, \theta}(1 + \varphi_u) = c \cdot \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1).$$

Proof. We write g and h for $1 + \varphi_u$ and $N'(1 + \varphi_u)$, respectively. Then, by Proposition 4.12, $h \in H^{\text{srs}}(F)$ is a norm of $g \in G^{\theta\text{-srs}}(F)$. Therefore we have $c \cdot \Theta_{\pi, \theta}(g) = \Theta_{\pi_H}(h)$ by Corollary 5.5.

On the other hand, since $N(g)$ is affine generic, g satisfies the assumption of Proposition 3.10. As the affine simple components of g are $(1, \dots, 1, u)$, we have

$$\Theta_{\pi, \theta}(1 + \varphi_u) = \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1).$$

\square

Corollary 5.7. *Let $\zeta \in \{\pm 1\}$. Let $\pi := \pi_{1, \zeta}$ and π_H its associated representation. Then, for $u \in k^\times$, we have*

$$\Theta_{\pi_H}(\varphi'_{u^2/2} N'(1 + \varphi_{u^2})) = c \cdot \Theta_{\pi, \theta}(\varphi_{u^2}(1 + \varphi_{u^2})) = c \cdot \zeta (\text{Kl}_{2^n u}^n(\psi) + \text{Kl}_{-2^n u}^n(\psi)).$$

Proof. We write g' and h' for $\varphi_{u^2}(1 + \varphi_{u^2})$ and $\varphi'_{u^2/2} N'(1 + \varphi_{u^2})$, respectively. Then, by Proposition 4.13, $h' \in H^{\text{srs}}(F)$ is a norm of $g' \in G^{\theta\text{-srs}}(F)$. Therefore we have $c \cdot \Theta_{\pi, \theta}(g') = \Theta_{\pi_H}(h')$ by Corollary 5.5.

On the other hand, since $N(g')$ is affine generic (see the proof of Proposition 4.13), g' satisfies the assumption of Proposition 3.10. As the affine simple components of $1 + \varphi_{u^2}$ are $(1, \dots, 1, u^2)$, we have

$$\Theta_{\pi, \theta}(\varphi_{u^2}(1 + \varphi_{u^2})) = \zeta (\text{Kl}_{2^n u}^n(\psi) + \text{Kl}_{-2^n u}^n(\psi)).$$

\square

5.2. Supercuspidality of π_H .

Proposition 5.8. *Let π be a self-dual simple supercuspidal representation of $G(F)$ with trivial central character. Let π_H be the irreducible smooth representation of $H(F)$ associated to π . Then π_H is supercuspidal.*

Proof. We use the theorem on a parametrization of supercuspidal representations of classical groups in [Xu15]. When viewed as a representation of W_F , ϕ is irreducible by the supercuspidality of π . Therefore the set $\text{Jord}(\phi_H)$ is a singleton $\{(\pi, 1)\}$ (see Section 2 in [Xu15] for the definition of the set $\text{Jord}(\phi_H)$). This satisfies the properties in [Xu15, Theorem 3.3], therefore π_H is supercuspidal. \square

5.3. Existence of I_H^{++} -fixed vector. Let $\zeta \in \{\pm 1\}$ and $\pi := \pi_{1,\zeta}$. Let π_H be the irreducible smooth representation of $H(F)$ associated to π .

Lemma 5.9. *Let $h \in H(F)$ be an affine generic element with its affine simple components $(h_1, \dots, h_n, h_{2n})$. Then $\Theta_{\pi_H}(h)$ is equal to either 0 or*

$$c \cdot \text{Kl}_{h_1 h_2^2 \dots h_n^2 h_{2n}/2}^{n+1}(\psi; 1, 2, \dots, 2, 1).$$

Proof. For such h , we take $g \in G(F)$ satisfying the conditions in Proposition 4.5. Since $h \in H^{\text{srs}}(F)$, $g \in G^{\theta\text{-srs}}(F)$, and h is a norm of g , we have

$$c \cdot \Theta_{\pi,\theta}(g) = \Theta_{\pi_H}(h)$$

by Corollary 5.5. We compute the left-hand side of this equality.

If there is not $x \in G(F)$ such that $xg\theta(x)^{-1} \in ZI^+\langle\varphi\rangle$, then $\Theta_{\pi,\theta}(g) = 0$ by the twisted character formula (Theorem 3.7).

Let us consider the case where there exists $x \in G(F)$ such that $xg\theta(x)^{-1} \in ZI^+\langle\varphi\rangle$. By Lemma 3.5, we may assume $x \in T(q)$. By replacing g with gz for some $z \in Z$, we may assume that $xg\theta(x)^{-1} \in I^+\langle\varphi\rangle$. Since $\theta(\varphi) = -\varphi^{-1}$ and φ normalizes I^+ , we have

$$(-1)^k \varphi^k xg\theta(\varphi^k x)^{-1} \in I^+ \amalg I^+ \varphi$$

for some $k \in \mathbb{Z}$. By replacing g with $(-1)^k g$ again, we may assume that

$$\varphi^k xg\theta(\varphi^k x)^{-1} \in I^+ \amalg I^+ \varphi.$$

If $\varphi^k xg\theta(\varphi^k x)^{-1}$ lies in $I^+ \varphi$, then

$$N(\varphi^k xg\theta(\varphi^k x)^{-1}) = \varphi^k xN(g)(\varphi^k x)^{-1} \in N(I^+ \varphi) \subset -I^+,$$

and this contradicts to the assumption that $N(g) \in I^+$. Therefore $\varphi^k xg\theta(\varphi^k x)^{-1} \in I^+$. Since the twisted character is invariant under θ -conjugacy, it suffices to compute

$$\Theta_{\pi,\theta}(\varphi^k xg\theta(\varphi^k x)^{-1}).$$

Let $x = \text{diag}(t_1, \dots, t_{2n})$. Let (g_1, \dots, g_{2n}) be the affine simple components of $\varphi^k xg\theta(\varphi^k x)^{-1}$. Then the affine simple components of $N(\varphi^k xg\theta(\varphi^k x)^{-1})$ are

$$(g_1 + g_{2n-1}, \dots, 2g_n, \dots, g_{2n-1} + g_1, 2g_{2n}).$$

On the other hand, since the affine simple components of $N(g)$ are

$$(h_2, \dots, h_n, h_n, \dots, h_1, 2h_{2n}),$$

those of $N(\varphi^k xg\theta(\varphi^k x)^{-1}) = \varphi^k xN(g)(\varphi^k x)^{-1}$ are the cyclic permutation of

$$\left(\frac{t_1}{t_2} h_2, \dots, \frac{t_{2n-1}}{t_{2n}} h_1, 2 \frac{t_{2n}}{t_1} h_{2n} \right).$$

Therefore $\varphi^k x g \theta(\varphi^k x)^{-1}$ satisfies the assumption of Proposition 3.10, and we have

$$\begin{aligned}\Theta_{\pi, \theta}(\varphi^k x g \theta(\varphi^k x)^{-1}) &= \text{Kl}_{g_n(g_1+g_{2n-1})^2 \cdots (g_{n-1}+g_{n+1})^2 g_{2n}}^{n+1}(\psi; 1, 2, \dots, 2, 1) \\ &= \text{Kl}_{h_1 h_2^2 \cdots h_n^2 h_{2n}/2}^{n+1}(\psi; 1, 2, \dots, 2, 1).\end{aligned}$$

□

Corollary 5.10. *The representation (π_H, V_H) has a nonzero I_H^{++} -fixed vector.*

Proof. We take $u \in k^\times$ such that $\text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1) \neq 0$ (such $u \in k^\times$ exists by Corollary A.5). Let $h := N'(1 + \varphi_u) \in I_H^+$ and $f := \mathbf{1}_{hI_H^{++}}$ the characteristic function of hI_H^{++} . Since the subgroup I_H^{++} is normal in I_H^+ , we have $I_H^{++}hI_H^{++} = hI_H^{++}$. Moreover hI_H^{++} is contained in $H^{\text{rs}}(F)$ by Proposition 4.5. Therefore f belongs to $\mathcal{H}(H, I_H^{++})$ and satisfies $\text{supp}(f) \subset H^{\text{rs}}(F)$.

By a property of the character (Theorem 3.1), we have

$$\text{tr } \pi_H(f) = \int_{H^{\text{rs}}(F)} f(y) \Theta_{\pi_H}(y) dy = \int_{hI_H^{++}} \Theta_{\pi_H}(y) dy.$$

For $y \in hI_H^{++}$, $\Theta_{\pi_H}(y)$ is equal to either $c \cdot \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1)$ or 0, by Lemma 5.9. Moreover we have

$$\Theta_{\pi_H}(h) = c \cdot \Theta_{\pi, \theta}(1 + \varphi_u) = c \cdot \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1) \neq 0$$

by Corollary 5.6. Since the character Θ_{π_H} is locally constant on $H^{\text{rs}}(F)$, we have

$$\int_{hI_H^{++}} \Theta_{\pi_H}(y) dy \neq 0.$$

On the other hand, by the definition of the distribution character,

$$\text{tr } \pi_H(f) = \text{tr } (\pi_H(f) | V_H^{I_H^{++}}).$$

Therefore we have $V_H^{I_H^{++}} \neq 0$. □

5.4. Simple supercuspidality of π_H . Let π and π_H be as in the previous subsection. In this subsection, we prove that π_H is simple supercuspidal.

By Corollary 5.10, the finite abelian group I_H^+/I_H^{++} acts on the nonzero subspace $\pi_H^{I_H^{++}}$ of π_H , and it decomposes into a direct sum of characters of I_H^+/I_H^{++} . We take a character η of I_H^+ which is contained in it.

Lemma 5.11. *If η is an affine generic character of I_H^+ , then π_H is simple supercuspidal.*

Proof. By the Frobenius reciprocity, we have

$$\text{Hom}_{H(F)}(\text{c-Ind}_{I_H^+}^{H(F)} \eta, \pi_H) \cong \text{Hom}_{I_H^+}(\eta, \pi_H|_{I_H^+}) \neq 0.$$

By Proposition 2.8 and Remark 2.7, the representation $\text{c-Ind}_{I_H^+}^{H(F)} \eta$ is a direct sum of two simple supercuspidal representations. Since π_H is irreducible, it is equivalent to one of them. □

From this lemma, we are reduced to prove the affine genericity of η . We first prove two technical lemmas which will be needed in the proof of the affine genericity of η .

Lemma 5.12. *Let $H_0 := \mathrm{SO}_{2n+1}(\mathcal{O})$ be a hyperspecial subgroup of $H(F)$. Let $y \in H(F)$. If y satisfies $yhy^{-1} \in H_0$ for an affine generic element $h \in H(F)$, then y belongs to $H_0 \amalg H_0\varphi'_1$.*

Proof. Let $y \in H(F)$ satisfying $yhy^{-1} \in H_0$ for an affine generic element $h \in I_H^+$. As affine genericity is preserved by I_H^+ -conjugation, any element of $H_0yI_H^+$ satisfies the same condition as y . It suffices to show $H_0yI_H^+ \subset H_0 \amalg H_0\varphi'_1$.

The isomorphism in Proposition 2.4 induces an isomorphism

$$H_0 \backslash H(F) / I_H^+ \cong ((N(F) \cap H_0) / T_1) \backslash (N(F) / T_1)$$

(see [HR08, Proposition 8]) and the right-hand side is represented by

$$T(\varpi^{\mathbb{Z}}) := \{ \mathrm{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, 1, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \mid r_1, \dots, r_n \in \mathbb{Z} \}.$$

Hence we may assume

$$\begin{aligned} y &= \mathrm{diag}(t_1, \dots, t_{2n+1}) \\ &= \mathrm{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, 1, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \in T(\varpi^{\mathbb{Z}}). \end{aligned}$$

Since $(yhy^{-1})_{ij} = t_i/t_j \cdot h_{ij}$, we have the following inequalities:

$$\begin{aligned} r_1 - r_2 &\geq 0, \\ &\vdots \\ r_{n-1} - r_n &\geq 0, \\ r_n &\geq 0, \text{ and} \\ -r_1 - r_2 &\geq -1. \end{aligned}$$

Therefore (r_1, \dots, r_n) is either $(0, \dots, 0)$ or $(1, 0, \dots, 0)$.

In the former case, we have $H_0yI_H^+ = H_0$.

In the latter case, we have

$$H_0yI_H^+ = H_0\varphi'_1I_H^+ = H_0I_H^+\varphi'_1 = H_0\varphi'_1$$

(recall that φ'_1 normalizes I_H^+). This completes the proof. \square

Lemma 5.13. *Let $\beta \in \Pi$ be a simple affine root of T_H in H . Then the subgroup $\langle I_H^{++}, U_\beta \rangle$ of $H(F)$ contains $H_{\mathbf{x}}^+$ for some point \mathbf{x} of the apartment $\mathcal{A}(H, T_H)$, where $H_{\mathbf{x}}^+$ is the pro-unipotent radical of the parahoric subgroup of $H(F)$ associated to \mathbf{x} .*

Proof. We recall that the pro-unipotent radical of the parahoric subgroup associated to a point $\mathbf{x} \in \mathcal{A}(H, T_H)$ and I_H^{++} are given by

$$\begin{aligned} H_{\mathbf{x}}^+ &= \langle T_1, U_\alpha \mid \alpha \in \Psi, \alpha(\mathbf{x}) > 0 \rangle, \text{ and} \\ I_H^{++} &= \langle T_1, U_\alpha \mid \alpha \in \Psi^+ \setminus \Pi \rangle, \end{aligned}$$

respectively. Therefore we have to take a point \mathbf{x} satisfying the following condition:

(*) If an affine root α satisfies $\alpha(\mathbf{x}) > 0$, then we have $\alpha \in (\Psi^+ \setminus \Pi) \cup \{\beta\}$.

We first consider the case where $\beta = -e_1 - e_2 + 1$. In this case, we take $\mathbf{x} = 0$. If $\alpha = a + r \in \Psi$ satisfies $\alpha(\mathbf{x}) > 0$, then we have $r > 0$. Hence $\alpha \in \Psi^+ \setminus \Pi$ or $\alpha = \beta$.

We next consider the case where $\beta = e_1 - e_2$. However, since $\varphi'_1 U_{e_1 - e_2} \varphi_1'^{-1} = U_{-e_1 - e_2 + 1}$, this case is reduced to the previous case.

We finally consider the other cases. We define a point $\mathbf{x} \in \mathcal{A}(H, T_H)$ as follows:

$$\mathbf{x} := \begin{cases} (\check{e}_1 + \cdots + \check{e}_i)/2 & (\beta = e_i - e_{i+1} \text{ for } 1 < i < n), \\ (\check{e}_1 + \cdots + \check{e}_n)/2 & (\beta = e_n). \end{cases}$$

It is enough to prove (*) for $\alpha = a + r_a \in \Psi$, where $r_a \in \mathbb{Z}$ is the smallest integer satisfying $a(\mathbf{x}) + r_a > 0$.

- If $a \in \Phi^+ \setminus \Delta$, then we have $0 \leq a(\mathbf{x}) \leq 1$. Hence $\alpha(\mathbf{x}) > 0$ implies that $r_a \geq 0$. Therefore $\alpha = a + r_a$ belongs to $\Psi^+ \setminus \Pi$.
- If $a \in \Delta$, then we have

$$a(\mathbf{x}) = \begin{cases} 0 & (a \neq \beta), \\ 1/2 & (a = \beta). \end{cases}$$

Hence we have

$$\alpha(\mathbf{x}) > 0 \iff \begin{cases} r_a > 0 & (a \neq \beta), \\ r_a \geq 0 & (a = \beta). \end{cases}$$

Therefore we have either $\alpha \in \Psi^+ \setminus \Pi$ or $\alpha = \beta$.

- If $a \in \Phi^- \setminus \{-e_1 - e_2\}$, then we have $-1 \leq a(\mathbf{x}) \leq 0$. Hence $\alpha(\mathbf{x}) > 0$ implies that $r_a \geq 1$. Therefore $\alpha = a + r_a$ belongs to $\Psi^+ \setminus \Pi$.
- If $a = -e_1 - e_2$, then we have $\alpha(\mathbf{x}) = r_a - 1$. Hence $\alpha(\mathbf{x}) > 0$ implies that $r_a \geq 2$, and we have $\alpha \in \Psi^+ \setminus \Pi$.

□

Proposition 5.14. *Every character η of I_H^+ which is contained in $\pi_H^{I_H^{++}}$ is affine generic. In particular, the representation π_H is simple supercuspidal.*

Proof. We suppose that η is not affine generic.

Since η is not affine generic, there exists a simple affine root $\beta \in \Pi$ such that its affine root subgroup U_β is contained in $\text{Ker}(\eta)$. Hence π_H has a nonzero $\langle I_H^{++}, U_\beta \rangle$ -fixed vector. Then, by Lemma 5.13, π_H also has a nonzero $H_{\mathbf{x}}^+$ -fixed vector for some point $\mathbf{x} \in \mathcal{A}(H, T_H)$, where $H_{\mathbf{x}}^+$ is the pro-unipotent radical of the parahoric subgroup associated to \mathbf{x} . Therefore the depth of π_H is zero.

On the other hand, π_H is generic with respect to any Whittaker datum. Indeed, as π is supercuspidal, ϕ is a bounded L -parameter. Hence ϕ_H is also bounded. Therefore the L -packet $\tilde{\Pi}_\phi$ of ϕ_H contains a generic representation by Proposition 8.3.2 in [Art13]. However $\tilde{\Pi}_\phi$ is a singleton consisting of π_H . Thus π_H is generic.

Then, by Lemma 6.1.2 in [DR09], the generic depth-zero supercuspidal representation π_H can be obtained by the compact induction of a representation of the reduction of a hyperspecial subgroup of $H(F)$. Since $H = \text{SO}_{2n+1}$ is an adjoint group, all hyperspecial subgroups of $H(F)$ are conjugate (see 2.5 in [Tit79]). Thus we may assume that the hyperspecial subgroup is $H_0 = \text{SO}_{2n+1}(\mathcal{O})$.

Let H_0^+ be the pro-unipotent radical of H_0 . Let $\pi_H \cong \text{c-Ind}_{H_0^+}^{H(F)} \dot{\rho}$, where $\dot{\rho}$ is the inflation of a representation ρ of $H_0/H_0^+ \cong \text{SO}_{2n+1}(k)$ to H_0 . Then, by the character formula (Theorem 3.2), we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = \sum_{\substack{y \in H_0 \setminus H(F) \\ yN'(1 + \varphi_u)y^{-1} \in H_0}} \text{tr}(\dot{\rho}(yN'(1 + \varphi_u)y^{-1}))$$

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for any $u \in k^\times$. By Lemma 5.12, we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = \text{tr}(\dot{\rho}(N'(1 + \varphi_u))) + \text{tr}(\dot{\rho}(\varphi'_1 N'(1 + \varphi_u) \varphi'_1{}^{-1})).$$

As

$$t \varphi'_1 N'(1 + \varphi_u) \varphi'_1{}^{-1} t^{-1} = N'(1 + \varphi_u),$$

where $t = \text{diag}(2/u, 1, \dots, 1, u/2) \in H_0$, we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = 2 \text{tr}(\dot{\rho}(N'(1 + \varphi_u))).$$

The image of $N'(1 + \varphi_u)$ in H_0/H_0^+ is independent of u , hence we can conclude that $\Theta_{\pi_H}(N'(1 + \varphi_u))$ is independent of u . However we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = c \cdot \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1)$$

by Corollary 5.6, and this is not constant on u by Corollary A.5. This is a contradiction. \square

5.5. Main theorem.

Proposition 5.15. *Let $a \in k^\times$, $\zeta \in \{\pm 1\}$, and $\pi := \pi_{a,\zeta}$. Then the representation π_H associated to π is given by $\pi'_{a/2,\zeta}$.*

Proof. By replacing the fixed uniformizer ϖ , we may assume that $\pi = \pi_{1,\zeta}$. By Proposition 5.14, π_H is simple supercuspidal. Let π_H be $\pi'_{b,\xi}$, where $b \in k^\times$ and $\xi \in \{\pm 1\}$. Our task is to determine b and ξ .

First, we consider b . For $u \in k^\times$, we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = c \cdot \Theta_{\pi,\theta}(1 + \varphi_u) = c \cdot \text{Kl}_{2^{2(n-1)}u}^{n+1}(\psi; 1, 2, \dots, 2, 1).$$

by Corollary 5.6. On the other hand, since $N'(1 + \varphi_u) \in I_H^+$ is an affine generic element with its affine simple components $(2, \dots, 2, u)$, we have

$$\Theta_{\pi_H}(N'(1 + \varphi_u)) = \text{Kl}_{2^{2(n-1)+1}bu}^{n+1}(\psi; 1, 2, \dots, 2, 1)$$

by Proposition 3.15. Hence $b = 1/2$ and $c = 1$ by Proposition A.6.

Next, we consider ξ . For $u \in k^\times$, we have

$$\Theta_{\pi_H}(\varphi'_{u^2/2} N'(1 + \varphi_{u^2})) = \Theta_{\pi,\theta}(\varphi_{u^2}(1 + \varphi_{u^2})) = \zeta (\text{Kl}_{2^{2n}u}^n(\psi) + \text{Kl}_{-2^{2n}u}^n(\psi))$$

by Corollary 5.7. On the other hand, $\varphi'_{u^2/2} N'(1 + \varphi_{u^2})$ is strongly regular semisimple (by Proposition 4.13) and the affine simple components of $(\varphi'_{u^2/2} N'(1 + \varphi_{u^2}))^2 \in I_H^+$ are given by $(4, \dots, 4, 2u^2)$. Hence $\varphi'_{u^2/2} N'(1 + \varphi_{u^2})$ satisfies the assumption of Proposition 3.19, and we have

$$\Theta_{\pi_H}(\varphi'_{u^2/2} N'(1 + \varphi_{u^2})) = \xi (\text{Kl}_{2^{2n}u}^n(\psi) + \text{Kl}_{-2^{2n}u}^n(\psi)).$$

By summing up over $u \in k^\times$, we have

$$2\zeta \sum_{u \in k^\times} \text{Kl}_u^n(\psi) = 2\xi \sum_{u \in k^\times} \text{Kl}_u^n(\psi).$$

Since $\sum_{u \in k^\times} \text{Kl}_u^n(\psi) = (-1)^n \neq 0$ by Proposition A.4, we have $\xi = \zeta$. \square

In summary, we get the following result.

Theorem 5.16 (Main theorem). *Let $b \in k^\times$ and $\xi \in \{\pm 1\}$. Under the parametrizations as in Sections 2.3 and 2.4, we have the following:*

- (1) *The L -packet containing the simple supercuspidal representation $\pi'_{b,\xi}$ of $\text{SO}_{2n+1}(F)$ is a singleton. In particular, the character of $\pi'_{b,\xi}$ is stable.*

- (2) The lifting of the simple supercuspidal representation $\pi'_{b,\xi}$ of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ is again simple supercuspidal, and given by $\pi_{2b,\xi}$.

Remark 5.17. From this result, we know that the L -parameter of $\pi'_{b,\xi}$ is equal to that of $\pi_{2b,\xi}$. On the other hand, L -parameters of simple supercuspidal representations of general linear groups have been determined explicitly by the works of [BH05] and [IT15]. Therefore we can get an explicit description of L -parameters of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$.

APPENDIX A. GAUSS SUM AND KLOOSTERMAN SUM

Definition A.1 (Gauss sum). Let $\chi: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ be a multiplicative character and $\psi: \mathbb{F}_q \rightarrow \mathbb{C}^\times$ a nontrivial additive character. We define the Gauss sum with respect to (χ, ψ) by

$$G(\chi, \psi) := \sum_{t \in \mathbb{F}_q^\times} \chi(t) \psi(t).$$

The following properties are well-known.

Proposition A.2. (1) If χ is trivial, then $G(\chi, \psi) = -1$.

(2) If χ is nontrivial, then $|G(\chi, \psi)| = \sqrt{q}$.

In particular, the sum $G(\chi, \psi)$ is nonzero.

Definition A.3 (Kloosterman sum, [Kat88]). Let $\psi: \mathbb{F}_q \rightarrow \mathbb{C}^\times$ be a nontrivial additive character, a an element of \mathbb{F}_q , and b_1, \dots, b_n positive integers. We define the Kloosterman sum with respect to $(\psi, a, b_1, \dots, b_n)$ by

$$\mathrm{Kl}_a^n(\psi; b_1, \dots, b_n) := \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q \\ x_1^{b_1} \cdots x_n^{b_n} = a}} \psi(x_1 + \cdots + x_n).$$

When $b_1 = \cdots = b_n = 1$, we denote it simply by $\mathrm{Kl}_a^n(\psi)$.

Proposition A.4. For a nontrivial additive character ψ , positive integers (b_1, \dots, b_n) , and a multiplicative character $\chi: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$, the following properties hold:

(1)

$$\mathrm{Kl}_0^n(\psi; b_1, \dots, b_n) = (-1)^{n-1},$$

(2)

$$\sum_{a \in \mathbb{F}_q} \mathrm{Kl}_a^n(\psi; b_1, \dots, b_n) = 0, \text{ and}$$

(3)

$$\sum_{a \in \mathbb{F}_q^\times} \chi(a) \mathrm{Kl}_a^n(\psi; b_1, \dots, b_n) = \prod_{i=1}^n G(\chi^{b_i}, \psi).$$

Proof. (1) By the inclusion-exclusion principle,

$$\begin{aligned} \mathrm{Kl}_0^n(\psi; b_1, \dots, b_n) &= \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q \\ x_1^{b_1} \cdots x_n^{b_n} = 0}} \psi(x_1 + \cdots + x_n) \\ &= \sum_{\substack{I \subset \{1, \dots, n\} \\ I \neq \emptyset}} \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q \\ x_i = 0, i \in I}} (-1)^{|I|-1} \psi(x_1 + \cdots + x_n) \\ &= (-1)^{n-1}. \end{aligned}$$

(2) We have

$$\begin{aligned}\sum_{a \in \mathbb{F}_q} \text{Kl}_a^n(\psi; b_1, \dots, b_n) &= \sum_{x_1, \dots, x_n \in \mathbb{F}_q} \psi(x_1 + \dots + x_n) \\ &= \left(\sum_{t \in \mathbb{F}_q} \psi(t) \right)^n \\ &= 0.\end{aligned}$$

(3) We have

$$\begin{aligned}\sum_{a \in \mathbb{F}_q^\times} \chi(a) \text{Kl}_a^n(\psi; b_1, \dots, b_n) &= \sum_{x_1, \dots, x_n \in \mathbb{F}_q^\times} \chi(x_1^{b_1} \dots x_n^{b_n}) \psi(x_1 + \dots + x_n) \\ &= \prod_{i=1}^n \sum_{x_i \in \mathbb{F}_q^\times} \chi(x_i^{b_i}) \psi(x_i) \\ &= \prod_{i=1}^n G(\chi^{b_i}, \psi).\end{aligned}$$

□

Corollary A.5. (1) *The sum $\text{Kl}_a^n(\psi; b_1, \dots, b_n)$ is not constant on $a \in \mathbb{F}_q^\times$.*
(2) *For some $a \in \mathbb{F}_q^\times$, $\text{Kl}_a^n(\psi; b_1, \dots, b_n) \neq 0$*

Proof. Assume that the sum $\text{Kl}_a^n(\psi; b_1, \dots, b_n)$ is constant on $a \in \mathbb{F}_q^\times$. If we take a nontrivial multiplicative character χ , then by Proposition A.4 (3) we have

$$\prod_{i=1}^n G(\chi^{b_i}, \psi) = 0.$$

However this contradicts Proposition A.2. □

Proposition A.6 ([IT14, Lemma 3.4]). *Let $a, b \in \mathbb{F}_q^\times$ and $c \in \mathbb{C}^\times$. If*

$$\text{Kl}_{ta}^n(\psi; b_1, \dots, b_n) = c \text{Kl}_{tb}^n(\psi; b_1, \dots, b_n)$$

for every $t \in \mathbb{F}_q^\times$, then $c = 1$ and $a = b$.

Proof. By summing up over $t \in \mathbb{F}_q^\times$ and using Proposition A.4, we get $(-1)^n = c(-1)^n$, therefore $c = 1$.

We next show $a = b$. We may assume $b = 1$. It suffices to show that if $a \neq 1$, there exists $t \in \mathbb{F}_q^\times$ such that $\text{Kl}_{ta}^n(\psi; b_1, \dots, b_n) \neq \text{Kl}_t^n(\psi; b_1, \dots, b_n)$. Let us assume $a \neq 1$, then we can take a multiplicative character χ satisfying $\chi(a) \neq 1$. Then we have

$$\begin{aligned}&\sum_{t \in \mathbb{F}_q^\times} \chi(t) (\text{Kl}_{ta}^n(\psi; b_1, \dots, b_n) - \text{Kl}_t^n(\psi; b_1, \dots, b_n)) \\ &= (\chi(a)^{-1} - 1) \sum_{t \in \mathbb{F}_q^\times} \chi(t) \text{Kl}_t^n(\psi; b_1, \dots, b_n) \\ &= (\chi(a)^{-1} - 1) \prod_{i=1}^n G(\chi^{b_i}, \psi) \\ &\neq 0.\end{aligned}$$

Hence $\mathrm{Kl}_{ta}^n(\psi; b_1, \dots, b_n) \neq \mathrm{Kl}_t^n(\psi; b_1, \dots, b_n)$ for some $t \in \mathbb{F}_q^\times$. \square

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